DISCUSSION PAPER SERIES

IZA

IZA DP No. 1239

Rationalizing Boundedly Rational Choice: Sequential Rationalizability and Rational Shortlist Methods

Paola Manzini
Marco Mariotti

August 2004

Forschungsinstitut
zur Zukunft der Arbeit
Institute for the Study
of Labor

Rationalizing Boundedly Rational Choice: Sequential Rationalizability and Rational Shortlist Methods

Paola Manzini
Queen Mary, University of London
and IZA Bonn

Marco Mariotti
Queen Mary, University of London

Discussion Paper No. 1239
August 2004

IZA

P.O. Box 7240
53072 Bonn
Germany

Phone: +49-228-3894-0
Fax: +49-228-3894-180
Email: iza@iza.org

IZA Discussion Paper No. 1239
August 2004

ABSTRACT

Rationalizing Boundedly Rational Choice: Sequential Rationalizability and Rational Shortlist Methods*

A *sequentially rationalizable choice function* is a choice function which can be obtained by applying sequentially a fixed set of asymmetric binary relations (*rationales*). A Rational ShortlistMethod (RSM) is a choice function which is sequentially rationalizable by two rationales. These concepts translate into economic language some human choice heuristics studied in psychology. We provide a full characterization of RSMs and study some properties of sequential rationalizability. These properties allow some degree of menu dependence in choice.

JEL Classification: D0

Keywords: rationalizability of choice, bounded rationality, intransitive choice, incomplete preferences, menu dependence

Corresponding author:

Paola Manzini
Department of Economics
Queen Mary
University of London
Mile End Road
London E 1 4 NS
United Kingdom
Email: p.manzini@qmul.ac.uk

* Most of this work was carried out while we were visiting Bocconi University in Milan. Their generous hospitality and financial support through fellowships are gratefully acknowledged. We thank Walter Bossert and Efe Ok for comments. The responsibility for any error is our own.

1 Introduction

In economics, to rationalize choice behavior means to look for a rationale (namely a preference relation) such that the choices are maximal according to that rationale across the entire domain of choice. Psychologists and students of bounded rationality, however, tend to model choices in ways that depart from the standard revealed preference model in (at least) two fundamental ways. First, choices are often seen as the result of the application of *multiple* criteria or rationales. Second, the rationales are applied *sequentially* rather than simultaneously, eliminating 'inferior' alternatives in successive steps, until only one alternative is left. This is natural when limited mental resources favor focussing, during the decision process, on just one rationale at a time.[1]

In this paper we aim to explain, using the conventional tools of economic theory rather than the language of psychology, choice behavior by means of sequentially applied criteria. At first blush it might seem that with a sufficiently large number of criteria any conceivable choice pattern could be explained. To the contrary, we find that explaining choices in this way provides a concept of rationality that poses a strong, though not obvious, demarcation between rational and irrational behavior. Compared with the classical axioms of revealed preference theory, our axioms allow several forms of *menu-dependence.* This means that revealing a preference for one alternative over another in some choice set is not equivalent to revealing the same preference in another set including both alternatives. That is, the condition of *binariness* of choice is violated[2] though not, as we will show, in arbitrary ways.

To be more precise, our first contribution is to introduce the concept of a *sequentially rationalizable choice function.* This is a (single valued) choice function for which there exists a fixed *sequence* of asymmetric binary relations (the *rationales*) that allows the retrieval of the choice from each feasible set by means of the following procedure. Take the maximal set according to the first relation. If this is a singleton, then this is the choice. Otherwise, take the maximal set according to the second relation applied to the previous maximal set. If this is a singleton, then this is the choice. Otherwise continue applying rationales in this fashion until a singleton maximal set is found. Note well: the *same* sequence of rationales is applied to each and every possible choice set.

Our idea of sequential rationalizability is thus in line with the general notion of eco-

[1]The best known early 'elimination heuristic' is Tversky's [23] Elimination By Aspects model. More recently, Gigerenzer and Todd [7] have promoted the general concept of 'fast and frugal' heuristics. Lexicographic choice heuristics are also referred to as non-compensatory choice heuristics in the psychological literature, to distinguish them from utility based approaches where a single indicator summarizes all characteristics of any given alternative, so that better and worse features of it compensate each other.

[2]See Nehring [18] and Sen [21] for a formalization and thorough discussion of menu-dependence and binariness. Their focus is however different from that of this paper.

nomic rationality as '*consistency*' across decision situations, even though it may violate binariness and even produce very 'strange' choice patterns, including cyclical ones. However not *every* violation of binariness is allowed for sequentially rationalizable choice. We will show which restrictions of menu-independence conditions are implied by sequential rationalizability.

Observe that there is no question of each single rationale being a complete relation. In fact the sequentiality of rationalization will normally allow, and in fact demand, rationales to be *incomplete* relations (corresponding to psychologists' 'fast and frugal' heuristics). In the standard revealed preference theory the fact that an alternative x is chosen from a set S while y is contained in S is taken as an expression of 'revealed preference' for x over y. This however assumes implicitly (1) that preferences are complete, and (2) that there is *only one* preference. If preferences are not necessarily complete or unique, what could the above choice pattern imply? For one, they could imply that while x might not necessarily be preferred to y, *something else* in S, say z, justifies the elimination of y at some stage, while x and z are not necessarily comparable at this stage. On the other hand, if we insist that a choice is made also between all alternatives, including x and z, it must be the case that x and z are compared using a *later* criterion than the one used to compare y and z.

For a concrete example, consider the case of an arbitrator who excludes from the possible partitions of an asset among claimants all those which are Pareto dominated (incomplete rationale). Then, he chooses among the remaining partitions one that he considers not to be dominated on fairness grounds. From this consistency perspective, the decision maker is rational: he employs sequentially a given set of criteria which is unchanging whatever the feasible set. However, he is not rational in the standard economic sense, since application of the procedure does not 'reveal' any single binary preference relation which is being maximized. In fact, this procedure may fail to satisfy the usual 'independence of irrelevant alternatives' property that if something is chosen from a large set then it must also be chosen from a smaller set. But is it unreasonable? Suppose that given the possible partitions A, B and C, A is Pareto-dominated by division B while no other Pareto comparisons can be made. So A will not be chosen from the grand set. However, why should A not be chosen were only A and C to be available? The first rationale to be used (Pareto) is now ineffective, and since A is fairer than C, the second rationale favors A. On the other hand, this story would be inconsistent with the arbitrator choosing A from a set in which B is present: so the rationalization in terms of the two criteria of Pareto and fairness does indeed restrict the extent of menu-dependence.

This example also illustrates another important point. Often just *two* criteria are enough to explain consistent menu dependent choices (even seemingly 'wild' ones, for example those that exhibit cycles). The arbitrator example mentioned above fits this description; other situations where choice can be made 'in two steps' are for instance:

- An investor can compare the expected returns of all portfolios, but she is unsure of how to rank all portfolios in terms of risk (although she can make some partial judgments). For her, risk is the primary consideration. So she focuses first on all portfolios which are not dominated in terms of risk. Then, she chooses among these the one with the highest expected return.

- A university hiring panel first discards all the candidates whose research falls in certain areas, and then chooses on the basis of publications.

To deal with this type of examples we define the notion of a '*Rational Shortlist Method*': the decision maker first identifies a shortlist of surviving alternatives based on some initial criterion, and then formulates his 'final' choice by selecting one of the shortlisted candidates using a second rationale. We will illustrate the power of rational shortlist methods by applying them in two contexts. We show how choice functions exhibiting certain types of time inconsistencies (preference reversal) and violations of von Neumann-Morgenstern independence in choice over lotteries (the ratio effect) are rational shortlist methods (see section 6).

The main result of this paper is a complete characterization of rational shortlist methods in terms of just two properties. One is the classical Expansion property (or property τ) requiring that when the choice from two distinct sets is the same, then it is also the choice from the union of the sets. The second is a weakening of the Weak Axiom of Revealed Preferences where we require that whenever some alternative is revealed preferred to another both in pairwise choices and in some other set, then in no subset can this revealed preference be overturned.

An interesting fact is that the *transitivity* or even the acyclicity of choice behavior is not at all a feature of the type of sequential consistency we study. Sequentially rationalizable choices, including rational shortlist methods, can well produce choice cycles. Indeed one of our results shows that (at least in the finite case) a choice function which is not standard rationalizable but is sequentially rationalizable involves pairwise cycles of choice.

This shows how sequential rationality is a substantial extension of traditional rationalization of choice functions. Yet, sequential rationalization is not a vacuous concept, in the sense that there exist choice functions which cannot be rationalized by sequential application of *any* number of rationales[3], including some well known procedures (e.g. 'Luce's dinner', 'choose the median' and 'second best' - see section 3) We identify some necessary conditions for sequential rationalizability by an arbitrary large but finite number of rationales. Finding sufficient conditions is still an open problem.

[3] So in this respect our approach is different from that in Kalai, Rubinstein and Spiegler [10].

2 Definitions

2.1 Preliminaries

Let X be a universal set of alternatives, with $|X| > 2$. Given a binary relation B on X (that is $B \subset X \times X$) its dual B^* is defined by $(x, y) \in B^* \Leftrightarrow (y, x) \in B$, and its complement B^C by $(x, y) \in B^C \Leftrightarrow (x, y) \notin B$. Given $S \subset X$ and a binary relation B denote the set of B-maximal elements of S by $\max(S; B) = \{x \in S| \text{ for no } y \in S \ (x, y) \in B^* \cap B^C\}$.

The *restriction* of $B \subset X \times X$ to a subset $Y \subset X$ is denoted $B|Y$.

The *transitive closure* of a binary relation B is denoted B^t and defined by $B^t = B \cup BB \cup BBB \cup ...$, that is to say xB^ty if and only if xBy or there exist $x_1, ..., x_n$ such that x_iBx_{i+1} for all $i = 1, ...n$ with $x_1 = x$ and $x_n = y$.

Let $\mathcal{P}(X)$ denote the set of all nonempty subsets of X.

2.2 Choice functions and 'revealed' relations

A *choice function* on X is a nonempty function $\gamma : \mathcal{P}(X) \to X$ with $\gamma(S) \in S$ for all $S \in \mathcal{P}(X)$.

Given a choice function γ, we use a number or 'revealed preference' relations. These distinctions are due to the fact that sequentially rationalizable choices will lack standard menu-independence and 'binariness' properties. So it will be perfectly possible for an alternative x to be 'revealed preferred' to another alternative y in some context (choice set), but for that preference to be reversed in another context, or even in pairwise comparisons.

$P_\gamma(S)$: *Revealed preference via* S. Write $(x, y) \in P_\gamma(S)$ whenever $x = \gamma(S)$ and $y \in S$.
P_γ: *Base relation.* Write $(x, y) \in P_\gamma$ whenever $x = \gamma(xy)$.
$P_\gamma^{BCons}(S) = P_\gamma(S) \cap P_\gamma$: *Binarily consistent revealed preference via* S.
$P_\gamma^{BInc}(S) = P_\gamma(S) \cap P_\gamma^*$: *Binarily inconsistent revealed preference via* S.
$\widetilde{P}_\gamma$: *Consistent binary revealed preference.* Write $(x, y) \in \widetilde{P}_\gamma$ whenever $(x, y) \in P_\gamma \cap P_\gamma^{*C}(S')$ for all $S' \in \mathcal{P}(X)$.

So $P_\gamma(S)$ denotes standard revealed preference via a set S, leaving open the issue of whether this preference will be confirmed in other choice sets, while P_γ is the *base relation.* On the other hand, P_γ^{BInc} and P_γ^{BInc} specify whether $P_\gamma(S)$ is or is not, respectively, consistent with pairwise choices. $\widetilde{P}_\gamma$ specifies that the base relation is not contradicted by choices in other sets, that is it relates x and y whenever x is revealed preferred to y in pairwise choices, and it never happens that y is revealed preferred to x in any other set.

Given $x \in X$ and $S \in \mathcal{P}(X)$ and a choice function γ, define the sets

$$Up_\gamma(x,S) = \{y \in S | (y,x) \in P_\gamma\}$$
$$Lo_\gamma(x,S) = \{y \in S | (x,y) \in P_\gamma\}$$

of elements of S which are chosen over x and over which x is chosen in pairwise choices, respectively.

Given $x \in X$, let $C_\gamma(x) = \{S \in \mathcal{P}(X) | x = \gamma(S)\}$. Note that the collection $\mathcal{C}_\gamma(x) = \{C_\gamma(x) \text{ for all } x \in X\}$ partitions $\mathcal{P}(X)$, that is $\bigcup_{C_\gamma(x) \in \mathcal{C}_\gamma(x)} C_\gamma(x) = \mathcal{P}(X)$, while $C_\gamma(x) \cap C_\gamma(y) = \emptyset$ for all $x, y \in X$ with $x \neq y$. In what follows, in order to describe a choice function we will sometimes use the γ notation and sometimes the C_γ notation according to expositional convenience.

2.3 Rationalizability

We now turn to the main concept introduced in the paper.

Definition 1 *A choice function γ is* ***sequentially rationalizable*** *whenever there exist asymmetric relations $P_1, ..., P_K \in X \times X$ such that, defining recursively*

$$M_0(S) = S$$
$$M_i(S) = \max(M_{i-1}(S); P_i), \; i = 1, ..., K$$

we have

$$\{\gamma(S)\} = M_K(S) \text{ for all } S$$

In that case we say that $P_1, ..., P_K$ sequentially rationalize γ. We call each P_i a **rationale**.

So the choice from each S can be constructed through sequential rounds of elimination of alternatives. At each round only the elements which are maximal according to a round-specific rationale survive. But, crucially, the rationales and the sequence are invariant with respect to the choice set,

Observe that while in the standard rationalization concept (via one relation) acyclicity of the rationalizing relation is necessary, since otherwise the nonemptiness of the maximal sets cannot be guaranteed, in this case acyclicity is necessary only for P_1, to guarantee that a maximal set exists in the initial round of elimination. The only other obvious necessary condition is that each rationale P_i is asymmetric (otherwise the rationalization of pairwise choices could not be guaranteed). But, of course, each of the rationales P_i will have to be acyclic on the surviving alternatives *at the moment of being applied*, that is when restricted to the sets $M_{i-1}(S)$.

Among the sequentially rationalizable choice functions, naturally focal are those which are 'closest' to the standard rationalizable ones, namely those which can be rationalized by just two rationales. They correspond to the standard method of selecting candidates (or alternatives) by first identifying a shortlist, with the additional requirement that the shortlisting must be consistent across choice sets in the sense of deriving from a single rationale, while the final choice is rationalized in the standard way over the sets of shortlisted candidates. So as a notable case of sequential rationalizability we have:

Definition 2 *A choice function γ is a **Rational Shortlist Method (RSM)** if it is sequentially rationalizable by two rationales.*

Finally, for future reference we list below some standard properties of choice functions:

Independence of Irrelevant Alternatives[4]: For all $S, T \in \mathcal{P}(X) : \gamma(T) \in S$, $S \subset T \Rightarrow \gamma(S) = \gamma(T)$.

Expansion: For all $x \in X : C_\gamma(x)$ is closed under set union.

WARP*:* $(x, y) \in P_\gamma(S)$ for some $S \in \mathcal{P}(X) \Rightarrow (y, x) \in P_\gamma^C(T)$ for all $T \neq S$, $T \in \mathcal{P}(X)$.

Condorcet: $(x, y) \in P_\gamma(S)$ for some $S \in \mathcal{P}(X) \Leftrightarrow (x, y) \in P_\gamma$.

Congruence: There does not exist a cycle $x_1 P_\gamma(S_1)\, x_2 P_\gamma(S_2) \ldots P_\gamma(S_{n-1})\, x_n P_\gamma(S_n)\, x_1$, $S_i \in \mathcal{P}(X)$ for all $i = 1, \ldots, n$.

Recall that, at least for the finite case, Condorcet is a necessary and sufficient condition for standard rationalizability, and Independence of Irrelevant Alternatives, which is equivalent to WARP, is a necessary and sufficient condition for standard rationalizability with an ordering. Congruence is in general equivalent to standard rationalizability. See e.g. Moulin [17] and Suzumura [22].

3 Some basic results on sequential rationalizability

The results in this section illuminate some features of the concept of sequential rationalizability. On the one hand, it helps rationalize choice functions which are far from rational in the standard sense (proposition 3). On the other hand, it still provides a clear demarcation between what is and what is not rational (proposition 4). Lastly, the third result (proposition 6) delimits the violations of standard rationalizability which sequential rationalizability can accommodate.

[4] For single-valued choice functions this conflates several properties of correspondences such as Chernoff's property ($S \subset T \Rightarrow \gamma(T) \cap \gamma(S) \subset \gamma(S)$) and Arrow's condition ($S \subset T$, $\gamma(T) \cap S \neq \emptyset \Rightarrow \gamma(S) = \gamma(T) \cap S$).

The example supporting our first result formalizes and extends the examples given in the introduction.

Proposition 3 *Let X be a finite set. Then there exist sequentially rationalizable choice functions γ on X such that P_γ is cyclic.*

Proof: Suppose first that $X = \{x, y, z\}$ and let γ be defined by

$$C_\gamma(x) = \{xy, xyz\}$$

$$C_\gamma(y) = \{yz\}$$

$$C_\gamma(z) = \{xz\}$$

so that the cycle $xP_\gamma yP_\gamma zP_\gamma x$ is generated. Define $P_1, P_2 \in X \times X$ by $P_1 = \{(y, z)\}$ and $P_2 = \{(x, y), (z, x)\}$. It is immediate to check that $\gamma(S) = \max(\max(S; P_1); P_2)$ for all $S \in \mathcal{P}(xyz)$.

For a finite X, fix any $x \in X$ and assume inductively that there exists a choice function γ on $\mathcal{P}(X \backslash x)$ which is sequentially rationalizable and which generates a base relation cycle, and let $P_1, ..., P_K$ be the rationalizing relations. Then extend γ to γ' on $\mathcal{P}(X)$ by setting $\gamma'(S) = \gamma(S \cap (X \backslash x))$ for all $S \neq \{x\}$, and let $P'_1, ..., P'_K \in X \times X$ be defined by $P'_i = P_i \cup \{(y, x) \,|\, y \in X \backslash x\}$. Obviously $P'_1, ..., P'_K$ sequentially rationalize γ'. ■

This implies in particular that sequentially rationalizable choice functions may violate Independence of Irrelevant Alternatives. We will show below that although, as we just proved, there may exist revealed binary preference cycles, the chosen element from a set must always be indirectly 'revealed preferred' to any other element through a sequence of binary choices.

Is the concept of sequential rationalizability vacuous? That is, is it possible to sequentially rationalize *every* choice function? Before answering this question, let us note that even if the answer was positive, the concept might still be of interest, because one could for example distinguish choice functions by the minimal number of rationales needed to rationalize them. This is the approach followed by Kalai, Rubinstein and Spiegler [10] whose rationalizability concept is vacuous in the above sense. However, for sequential rationalizability the answer is negative:

Proposition 4 *There exist choice functions which cannot be sequentially rationalized.*

Proof. Let $x, y, z \in X$ and let

$$xy, xz \in C_\gamma(x)$$

$$xyz, yz \in C_\gamma(y)$$

Suppose by contradiction that γ was sequentially rationalizable by $P_1, ..., P_K$. Let $i(x, y)$ be the smallest i such that some P_i relates x and y, that is

$$i(x, y) = \min\{i \in \{1, ..., K\} \mid (x, y) \in P_i \cup P_i^*\}$$

and similarly let

$$i(x, z) = \min\{i \in \{1, ..., K\} \mid (x, z) \in P_i \cup P_i^*\}$$

Since $xy \in C_\gamma(x)$ it must be $(x, y) \in P_{i(x,y)}$. Given this, $xyz \in C_\gamma(y)$ can only hold if $(z, x) \in P_{i(x,z)}$, which contradicts $xz \in C_\gamma(x)$. ■

Interestingly, the pattern of choice over the subsets of $\{x, y, z\}$ in the proof of Proposition 4 can be generated by several well-known procedures that have attracted economists' attention. Such procedures are therefore non rationalizable not only in the standard sense, but also in the weaker sense of this paper.

The first procedure is (a refinement of) the '*choose the median*' procedure defined as follows. There is a 'fundamental' order B on X (e.g. given by ideology from left to right) such that $xByBz$. The decision-maker chooses the median according to B, breaking ties by picking the highest element in the set of median elements.

Similarly, the choice pattern is consistent with the '*never choose the uniquely largest*' procedure. There is again a fundamental order B on alternatives and the chosen alternative cannot be the unique maximizer of B. However, to interpret the choice pattern in this way the fundamental ordering must be exactly the reverse of the one used for the choose the median procedure, namely $zByBx$. Baigent and Gaertner [2] and Gaertner and Xu ([5], [6]) have axiomatized these procedures.

A third procedure generating the choice pattern is the one described in the dinner example by Luce and Raiffa [11] (see also [10]). Imagine that when z is not available the decision maker chooses the greatest element according to the ordering B_1 given by xB_1y, while when z is available he chooses the greatest element according to the ordering B_2 given by yB_2xB_2z. This yields the sequentially non-rationalizable choice function. On the other hand, if the same procedure was followed but the ordering B_2 was given by yB_2zB_2x (which is also in the spirit of Luce and Raiffa's example), it would be possible to sequentially rationalize the choice function by applying first $P_1 = \{(z, x), (y, z)\}$ and then $P_2 = \{(x, y)\}$.

We now move to the question of what type of violations of standard rationalizability sequential rationalizability can encompass. A simple lemma will be useful in what follows.

Lemma 5 *Let γ be a sequentially rationalizable choice function. Then:*

$$(x, y) \in P_\gamma \text{ for all } y \in S \Rightarrow x = \gamma(S)$$

Proof. Let $i(x, y)$ be defined as in the proof of proposition 4. Let $(x, y) \in P_\gamma$ for all $y \in S$. For each $y \in S \backslash x$ we must have $(x, y) \in P_{i(x,y)}$, so that the successive application of the rationales eliminates all $y \in S \backslash x$, and no rationale can eliminate x. Therefore $x = \gamma(S)$, as desired. ■

To glean some intuition, consider a set $X = \{x, y, z\}$ with $\gamma(X) = x$. There are only three possible configurations of choice which violate Independence of Irrelevant Alternatives (hence are not standard rationalizable), namely one other alternative is chosen in pairwise choices over $x = \gamma(X)$: *(i)* $(y, x), (x, z), (z, y) \in P_\gamma$, *(ii)* $(y, x), (z, x) \in P_\gamma$, and *(iii)* $(y, x), (x, z), (y, z) \in P_\gamma$. Of these, only *(i)*, which exhibits a P_γ cycle, is sequentially rationalizable. In case *(ii)* x is never chosen in pairwise choices and so clearly the choice $x = \gamma(X)$ cannot be sequentially rationalized, while in case *(iii)* the alternative y is always chosen in pairwise comparisons when available, so by lemma 5 the only sequentially rationalizable choice from the grand set is $y = \gamma(X)$.

As it turns out, this is a general feature of sequentially rationalizable choice functions. That is, if γ over a finite set X violates Independence of Irrelevant Alternatives but is sequentially rationalizable, then γ generates pairwise cycles of choice.

Proposition 6 *Let X be finite and let γ be a sequentially rationalizable choice function on X. Then either γ satisfies Independence of Irrelevant Alternatives or P_γ is cyclic.*

Proof. Let $x = \gamma(X)$ be a sequentially rationalizable choice function that violates Independence of Irrelevant Alternatives. Then we can define $\mathcal{T} := \{T \subseteq X | T \ni x \neq \gamma(T)\}$ and let

$$S \in \arg\min_{T \in \mathcal{T}} |T|$$

be (one of) the smallest set(s) containing x but from which x is not chosen. It cannot be that $|S| > 2$, otherwise $(x, y) \in P_\gamma$ for all $y \in S$, and lemma 5 would imply $x = \gamma(S)$, contradicting the construction. Then it must be that $|S| = 2$. Let $S = \{x, y\}$. Since $x = \gamma(X)$, by lemma 5 there must exist some other alternative $z \in X$ such that $(z, y) \in P_\gamma$ in order for γ to be sequentially rationalizable. Now if $(x, z) \in P_\gamma$ we would have the cycle $y P_\gamma x P_\gamma z P_\gamma y$, and we would be done. So let $(z, x) \in P_\gamma$. Then there must exist some other alternative $w \in X$ such that $(w, z) \in P_\gamma$ in order for $x = \gamma(X)$ to be rationalized. If either x or y where to be chosen over w in pairwise comparisons this would generate cycles (either $xP_\gamma wP_\gamma zP_\gamma x$ or $yP_\gamma wP_\gamma zP_\gamma y$), so it must be that $(w, x), (w, y) \in P_\gamma$. Iterating this argument and noting that the set X is finite, however, either there exists some $a \neq x$ which is chosen over all others in pairwise choices, in which case $x = \gamma(X)$ cannot be rationalized; or there are cycles. ■

4 Rational Shortlist Methods: a complete characterization

In this section we focus on RSM's and prove the main characterization result of the paper.

We begin by observing that the property of Expansion would rule out the choice function used in the proof of proposition 4. This property will be shown below to be a necessary condition for a choice function to be an RSM. However there are domains and choice functions on those domains which satisfy Expansion but are not sequentially rationalizable by any number of rationales, as the following example proves.

Example 7 $X = \{x, y, w, z\}$

$C_\gamma(w) = \{wx, wy, wxy\}$

$C_\gamma(x) = \{xy, xz, xyz, wxyz\}$

$C_\gamma(y) = \{yz, wyz\}$

$C_\gamma(z) = \{wz, wxz\}$

Observe that each $C_\gamma(.)$ is closed under union and therefore this choice function satisfies Expansion. We now show that it cannot be rationalized. Let $i(x, y)$ be defined as in the proof of proposition 4. Then:

1. It cannot be $i(w, x) \leq i(z, w)$. If so, w would not have been eliminated when $P_{i(w,x)}$ is applied to X, so that $x = \gamma(X)$ would not be rationalized;

2. It cannot be $i(w, x) > i(z, w)$. If so, $z = \gamma(wxz)$ cannot be rationalized: if $i(x, z) \leq i(z, w)$ sequential application of the criteria to wxz would select $\gamma(wxz) = w$, whereas if $i(x, z) > i(z, w)$ sequential application of the criteria would select $\gamma(wxz) = x$.

The base relation P_γ for example 7 is visualized in figure 1, where $a \to b$ stands for $(a, b) \in P_\gamma$. Moreover, there are choice functions which satisfy Expansion but are not RSM's, although they are sequentially rationalizable, as shown in example 8.

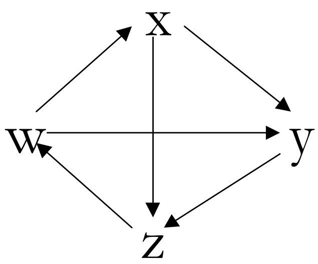

Figure 1: The base relation of example 7

Example 8 $X = \{x, y, w, z\}$

$C_\gamma(w) = \{wx\}$

$C_\gamma(x) = \{xy, xz, xyz, wxy, wxyz\}$

$C_\gamma(y) = \{wy, yz, wyz\}$

$C_\gamma(z) = \{wz, wxz\}$

The base relation P_γ is visualized in figure 2. It is straightforward to verify that this choice function is rationalized by $P_1 = \{(y, w)\}$, $P_2 = \{(z, w), (w, x), (x, y)\}$ and $P_3 = \{(x, z), (y, z)\}$. However it cannot be rationalized by just two rationales. To see this, suppose $(w, x) \in P_1$.Then $x = \gamma(wxyz)$ could not be rationalized. Suppose then that $(w, x) \in P_2$. Then $z = \gamma(wxz)$ cannot be rationalized, for x will eliminate z regardless of whether $(x, z) \in P_2$ or $(x, z) \in P_1$.

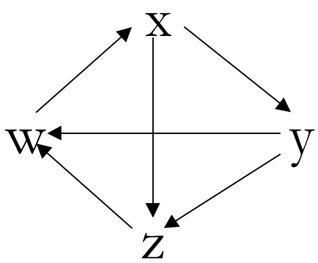

Figure 2: The base relation for example 8.

Motivated by this example, we introduce a property which is violated in it, and which we will show below to be another necessary condition for a choice function to be an RSM. The property is a weakening of WARP, in which the premise of the original axiom is strengthened (by adding the requirement that the alternative which is revealed preferred also dominates the inferior alternative in the base relation), and the conclusion is weakened (by restricting the set of choice sets for which the inferior alternatives cannot be in turn be revealed preferred).

Restricted WARP: $(x,y) \in P_{\gamma}^{BCons}(S)$ for some $S \in \mathcal{P}(X) \Rightarrow (y,x) \in P_{\gamma}^{C}(R)$ for all $R \subset S$.

In other words, if x is revealed preferred to y both in the pairwise comparison and in a larger set, then y cannot be revealed preferred to x in any 'intermediate' set in which x is present. The interpretation is straightforward: the pairwise preference for x over y does not exclude in principle that in larger sets some reason can be found to exclude x and choose y instead. However, if a set S does not contain any such reason, no smaller set can contain such a reason either. More vividly: if steak is chosen over chicken both when only steak and chicken are on the menu, and when a large selection of pizzas is also on the menu, then chicken will not be chosen when steak and a smaller selection of pizzas are available.

Example 8 does not satisfy Restricted WARP: in particular, we have that $x = \gamma(xz)$ and $x = \gamma(wxyz)$, so that it should not be possible that z is chosen when x is present. Instead, we have $z = \gamma(wxz)$ (the same applies to example 7). Restricted Warp is in fact some form of menu independence requirement which, while not excluding menu effects completely, confers on them a certain regularity.

The next result provides three necessary condition for a choice function to be an RSM. Two of these are Expansion and Restricted WARP, while the other is the revealed preference property below:

Restricted Condorcet: For all $S \in \mathcal{P}(X)$: (a) $x = \gamma(S) \Rightarrow (x,y) \in (P_{\gamma}|S)^{t}$ for all $y \in S$; (b) $(x,y) \in P_{\gamma}$ for all $y \in S \Rightarrow x = \gamma(S)$.

Part (b) of Restricted Condorcet is identical to the corresponding direction of Condorcet[5]. We have already shown in lemma 5 that this is a necessary condition for sequential rationalizability. Part (a) of this property says that the chosen alternative from a set must be *indirectly* revealed preferred in the set to any other alternative by using *pairwise* choices, that is via the base relation. In other words, for each unchosen alternative y in S there exists a sequence of alternatives in S, each of which is in the base relation with the next one, connecting the chosen alternative to y. This part is clearly a weakening of one direction of Condorcet, in which the direct pairwise revealed preference property has been replaced by indirect pairwise revealed preference. A choice function satisfying Restricted Condorcet but not Condorcet will exhibit some degree of 'menu dependence'.

[5]This is sometimes called condition γ. Nehring [18] studies menu-dependence by weakening condition γ.

Proposition 9 *Let γ be a Rational Shortlist Method. Then it satisfies (i) Expansion, (ii) Restricted Condorcet and (iii) Restricted WARP.*

Proof. Let γ be an RSM and let P_1 and P_2 be the rationales.

(i) Expansion. Let $x = \gamma(S) \cap \gamma(T)$ for $S, T \in \mathcal{P}(X)$. If for some $y \in S \cup T$ it were $(y, x) \in P_1$, this would immediately contradict $x = \gamma(S)$ or $x = \gamma(T)$ and γ being rationalized. Suppose now that for some $y \in M_1(S \cup T)$ we had $(y, x) \in P_2$. Since obviously $M_1(S \cup T) \subset M_1(S) \cup M_1(T)$, we have $y \in M_1(S)$ or $y \in M_1(T)$, contradicting $x \in M_2(S)$ or $x \in M_2(T)$.

Therefore for all $y \in S \cup T$ we have $(y, x) \in P_1^C \cup P_2^C$, x survives both rounds of elimination and we can conclude that $x = \gamma(S \cup T)$.

(ii) Restricted Condorcet. (We will show later that Restricted Condorcet is necessary for sequential rationalizability. Since an RSM is sequentially rationalizable, that result would suffice to prove (ii). However, we give here a direct proof of part (a), which is of interest in itself because it shows the connection between Restricted Condorcet and Expansion).

Let $x = \gamma(S)$. Let R be the subset of elements of S which are chosen over x in pairwise choices, that is $R = Up_\gamma(x, S)$. Let Q be the subset of elements of R to which x is pairwise indirectly revealed preferred in S, that is $Q = \left\{z \in R \mid (x, z) \in (P_\gamma|S)^t\right\}$. Since trivially $(x, y) \in (P_\gamma|S)^t$ for all $y \in Lo_\gamma(x, S)$, the assertion of the statement is proved if we show that $R \backslash Q$ is empty.

Suppose not, and let $y \in R \backslash Q$. Suppose that there existed $z \in Q$ with $(z, y) \in P_\gamma$. Since $(x, z) \in (P_\gamma|S)^t$ this would imply $(x, y) \in (P_\gamma|S)^t$, a contradiction. Therefore it must be $(y, z) \in P_\gamma$ for all $z \in Q$. A similar contradiction is obtained by supposing that there existed $z \in Lo_\gamma(x, S)$ such that $(z, y) \in P_\gamma$. Therefore $(y, z) \in P_\gamma$ for all $z \in Lo_\gamma(x, S)$.

Note now that S can be expressed as $S = \{x \cup Lo_\gamma(x, S) \cup Q \cup R \backslash Q\}$. We have just shown that $yz \in C_\gamma(y)$ for all $z \in Q \cup Lo_\gamma(x, S)$, and we have proved in *(i)* that γ satisfies Expansion, so that $Q \cup Lo_\gamma(x, S) \in C_\gamma(y)$. Moreover by assumption $xy \in C_\gamma(y)$ and $R \backslash Q \in C_\gamma(y)$. Applying Expansion again then we obtain the contradiction $S \in C_\gamma(y)$. We can conclude that $R \backslash Q$ is empty.

(iii) Restricted WARP. Let $(x, y) \in P_\gamma^{BCons}(S) = P_\gamma(S) \cap P_\gamma$. Then $(x, y) \in P_\gamma$ implies that $(x, y) \in P_1 \cup P_2$. If $(x, y) \in P_1$, then the desired conclusion follows immediately. Suppose then that $(x, y) \in P_2$. The fact that $(x, y) \in P_\gamma(S)$ implies that for all $z \in S$ it is the case that $(z, x) \in P_1^C$. Therefore $x \in M_1(R)$ for all $R \subset S$ for which $x \in R$. Since $(x, y) \in P_2$ then $y \notin M_2(R)$ for all such R, and thus $y \notin \gamma(R)$. ∎

Note that the argument in the proof of (i) could not be iterated further in the case of more than two rationales since it is not necessarily true that $M_2(S \cup T) \subset M_2(S) \cup$

$M_2(T)$. There could in fact be $y \in (M_1(S) \cup M_1(T)) \setminus M_1(S \cup T)$ such that $(y,z) \in P_2$ for some $z \in M_1(S) \cup M_1(T)$ while for all $y' \in M_1(S \cup T)$ it is the case that $(y',z) \in P_2^C$. So if it were $(z,x) \in P_3$, x could not be chosen from $S \cup T$.

We can now turn to our main result, which completely characterizes RSM's.

Theorem 10 *A choice function γ is a Rational Shortlist Method if and only if it satisfies Expansion and Restricted WARP*

Proof. In view of Proposition 9 we only need to prove sufficiency. Suppose that γ satisfies the axioms. We will explicitly construct the rationales.

For all $S \in \mathcal{P}(X)$ such that for some $x, z \in S$ we have $(x,z) \in P_\gamma^{BInc}(S)$, define

$$Y(z,S) = \left\{ y \in S | \, (y,z) \in \widetilde{P}_\gamma \right\}$$

Next define the following relations:

$$P_2 = \left\{ (z,x) \in X \times X | \exists S \in \mathcal{P}(X) \text{ with } (x,z) \in P_\gamma^{BInc}(S) \right\}$$

$$P_1' = \left\{ (y,z) \in X \times X | \exists S \in \mathcal{P}(X) \text{ with } (x,z) \in P_\gamma^{BInc}(S) \text{ and } y \in Y(z,S) \right\}$$

$$P_1'' = \{ (x,y) \in (X \times X) \setminus (P_1' \cup P_2) \,|\, (x,y) \in P_\gamma \}$$

$$P_1 = P_1' \cup P_1''$$

In words: P_2 relates z to x whenever x is chosen from a set S but z in S is chosen over x in pairwise comparisons. P_1 is constructed by joining two relations P_1' and P_1'' where: P_1' relates y and z whenever there are x and z related by P_2, y is chosen over z in pairwise comparison, and in addition it never happens that z is chosen from a set in which y is present. The pairs which have been left unallocated are related by P_1'' in a manner consistent with pairwise choices.

All the relations defined are asymmetric, since if $(x,y) \in P_i$, $i \in \{1,2\}$, then $(x,y) \in P_\gamma$ and P_γ is asymmetric. Also observe that $P_2 \cap P_1 = \emptyset$. In fact, if $(x,y) \in P_2$ then there exists $T \in \mathcal{P}(X)$ such that $(x,y) \in P_\gamma^{BInc}$, which prevents $(x,y) \in P_1'$; and obviously $P_2 \cap P_1'' = \emptyset$. Finally note that P_1 is acyclic. For suppose to the contrary that we have a cycle $x_1 P_1 x_2 P_1 ... P_1 x_n P_1 x_1$. Let without loss of generality $x_1 = \gamma(x_1 x_2 ... x_n)$. But then by definition of P_1 it cannot be $(x_n, x_1) \in P_1$.

Next, we show that the set $Y(z,S)$ in the definition of P_1' is never empty. This means that for each $S \in \mathcal{P}(X)$ with $(x,z) \in P_\gamma^{BInc}(S)$ it is always possible to find $y \in S$ such that $(y,z) \in P_\gamma|S$ and such that in addition $(z,y) \in P_\gamma^C(T)$ for all $T \in \mathcal{P}(X)$. Begin by noting that the set $Up_\gamma(z,S)$ is nonempty (to see this, suppose in contradiction that $(z,w) \in P_\gamma$ for all $w \in S \backslash z$. Then z cannot be eliminated either via P_1 or via P_2 and the choice $x = \gamma(S)$ could not be rationalized). Suppose by contradiction that for every $y \in Up_\gamma(z,S)$ there exists a $T_y \in \mathcal{P}(X)$ such that $(z,y) \in P_\gamma(T_y)$. Let $T = \bigcup_{y \in Up_\gamma(z,S)} T_y$. Since $(z,x) \in P_\gamma$ by assumption and $(z,y') \in P_\gamma$ for all $y' \in S \backslash Up_\gamma(z,S)$ by definition, Expansion implies that $z = \gamma(S \cup T)$. But then $x = \gamma(S)$ contradicts Restricted WARP.

We can now check that P_1 and P_2 thus defined rationalize γ. Take any $S \in \mathcal{P}(X)$ and let $x = \gamma(S)$. All $z \in S$ such that $(z,x) \in P_\gamma$ are eliminated by the application of P_1 (since we showed that $Y(x,z,S)$ is nonempty for all S), while x is not eliminated by P_1 as it is related to all such z only by P_2 and not by P_1. Therefore x must survive both the first and the second round of elimination. For all other $y \in S$ we have $(x,y) \in P_\gamma$, and either there exists $T \in \mathcal{P}(X)$ such that $(y,x) \in P_\gamma(T)$, or not. In the former case we have $(x,y) \in P_2$ and in the latter case we have $(x,y) \in P_1'$. Therefore all such y are eliminated either in the first or in the second round. ■

The above Theorem relates sequential rationalizability to some classical rationality properties. However, the proof makes it clear that a weaker single requirement also characterizes an RSM, namely:

Neutralization: For all $S \in \mathcal{P}(X)$: $(x,z) \in P_\gamma^{BInc}(S) \Rightarrow \exists y \in S : (y,z) \in \widetilde{P}_\gamma$.

In words, if x is revealed preferred to z in some set S, but z is directly revealed preferred to x in pairwise choice, then there exists some other alternative y which 'neutralizes' z, in the sense that it is chosen over it in pairwise comparisons *and* it is never revealed dispreferred to z. We can then state the following:

Corollary 11 *A choice function γ is a Rational Shortlist Method if and only if it satisfies Neutralization.*

Proof. Sufficiency follows directly from the proof of Theorem 10. For necessity, suppose P_1 and P_2 sequentially rationalize γ. Suppose that $(x,z) \in P_\gamma^{BInc}(S)$. Then it must be that $(z,x) \in P_2$. Moreover, to rationalize $x = \gamma(S)$ there must exist $y \in S$ such that $(y,z) \in P_1$. This implies that there does not exist $S' \in \mathcal{P}(X)$ such that $(z,y) \in P_\gamma(S')$, and therefore Neutralization must hold. ■

In the sequel of this section we offer some observations regarding the properties that the rationales of an RSM may satisfy in addition to asymmetry of both and acyclicity of P_1. Although for RSM's the acyclicity of P_2 cannot be guaranteed, the weaker property of 3-acyclicity (see e.g. Suzumura[22]) can be established. A binary relation B on $X \times X$ is *3-acyclic* if $(x,y),(y,z) \in B$ implies $(z,y) \in B^C$ for any three alternatives $x,y,z \in X$.

Corollary 12 *Let γ be a Rational Shortlist Method rationalized by P_1 and P_2. Then P_2 is 3-acyclic.*

Proof. Suppose not. then by the definition of P_2 there must exist sets S_i and alternatives x_i, $i = 1, 2, 3$, for which $(x_i, x_{i-1}) \in P_\gamma^{BInc}(S_i)$ for $1 < i \leq 3$ and $(x_1, x_3) \in P_\gamma^{BInc}(S_1)$. Without loss of generality, let $x_1 = \gamma(x_1x_2x_3)$. By Proposition 9 γ satisfies Expansion and Restricted WARP. By Expansion we have that $x_3 = \gamma(S_3 \cup x_1)$. This together with $(x_3, x_1) \in P_\gamma$and $x_1x_2x_3 \subset S_3 \cup x_1$ implies by Restricted WARP that $(x_1, x_3) \in P_\gamma^C(x_1x_2x_3)$, a contradiction. ■

The following axiom, derived form the classical Congruence property, will guarantee the full acyclicity of P_2.

Restricted Congruence: $x_1P_\gamma(S_1)\,x_2P_\gamma(S_2)\,...P_\gamma(S_{n-1})\,x_nP_\gamma(S_n)\,x_1,\ S_i \in \mathcal{P}(X)$ for all $i = 1, ..., n \Rightarrow (x_{i+1}, x_i) \in P_\gamma^C$ for some $i < n - 1$ or $(x_n, x_1) \in P_\gamma^C$.

Restricted Congruence allows revealed preference cycles to exist. However, it requires that in that case there should not exist a base relation cycle which goes in the opposite direction to the revealed preference cycle.

Corollary 13 *A choice function γ is a Rational Shortlist Method for which both rationales are acyclic if and only if γ satisfies Expansion, Restricted WARP and Restricted Congruence.*

Proof. Follows from Theorem 10. Restricted Congruence is immediately seen to be equivalent to prohibiting P_2 cycles with the definition of P_2 given in the proof of Theorem 10, which proves sufficiency. Conversely, suppose that Restricted Congruence is violated for some Rational Shortlist Method γ. That is, suppose that

$$x_1P_\gamma(S_1)\,x_2P_\gamma(S_2)\,...P_\gamma(S_{n-1})\,x_nP_\gamma(S_n)\,x_1,\ S_i \in \mathcal{P}(X) \text{ for all } i = 1, ..., n$$

and yet $(x_{i+1}, x_i), (x_n, x_1) \in P_\gamma$ for all $i < n - 1$. It is clear that *any* rationalization with two rationales P_1 and P_2 will be such that $(x_{i+1}, x_i), (x_n, x_1) \in P_2$ for all $i < n - 1$ (otherwise $x_i = \gamma(S_i)$ for all i could not be rationalized). Therefore P_2 would be cyclic.■

Clearly if P_2 is acyclic, it can also be chosen to be transitive, simply by taking its transitive closure. This however would be rather artificial, as it would include pairs which contradict the base relation P_γ.

One may wonder whether the P_1 rationale may be chosen to satisfy, beyond acyclicity, also transitivity. We conclude this section with an example of a choice function which satisfies Expansion and Restricted WARP (and therefore by Theorem 10 is an RSM) but which cannot be sequentially rationalized using a transitive P_1.

Example 14 $X = \{x, y, w, z\}$

$C_\gamma(w) = \{wz\}$

$C_\gamma(x) = \{xz, wx, xyz, wxz, wxyz\}$

$C_\gamma(y) = \{xy, wy, wxy\}$

$C_\gamma(z) = \{yz, wyz\}$

Since γ satisfies Expansion and Restricted WARP it is an RSM. Because $xyz \in C_\gamma(x)$, $xy \in C_\gamma(y)$ and $yz \in C_\gamma(z)$ we must have $(z, y) \in P_1$. Because $wy \in C_\gamma(y)$, $wz \in C_\gamma(w)$ and $wyz \in C_\gamma(z)$ we must have $(y, w) \in P_1$. Transitivity of P_1 then would imply $(z, w) \in P_1$. Therefore $(z, w) \in P_\gamma$, contradicting $C_\gamma(w) = \{wz\}$ and the rationalizability of γ.

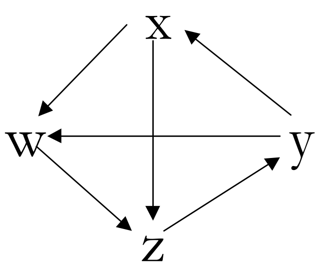

Figure 3: The base relation for example 14.

5 Sequential rationalizability: Some necessary conditions

In this section we look for patterns of choice behavior that cannot be sequentially rationalized. Since sequential rationalizability allows menu-dependence, one way to look at the results of this section is as a characterization of the limits that sequential rationalizability imposes on menu dependence.

A comparison between Example 7 and Example 8 motivates the following property, which is violated in Example 7 but not in Example 8. Recall that, given $x \in X$ and $S \in \mathcal{P}(X)$, $Up_\gamma(x, S) = \{y \in S | (y, x) \in P_\gamma\}$. We define a new type of revealed relation by writing $(x, y) \in \widehat{P}_\gamma$ whenever $(x, z) \in P_\gamma$ for all $z \in Up_\gamma(y, S)$. In words, $\widehat{P}_\gamma$ relates x and y whenever x is directly revealed preferred to anything that is directly revealed preferred to y.

Menu: For all $S \in \mathcal{P}(X)$: $(x, y) \in P_\gamma^{BInc}(S) \cap \widehat{P}_\gamma \Rightarrow x = \gamma(xyz)$ for some $z \in S \backslash xy$.

Menu is a contraction type property that represents a substantial weakening of Independence of Irrelevant Alternatives. The premise of the axiom specifies a certain form of menu dependence. Although y is revealed preferred to x in pairwise choices, x is revealed preferred to y both directly in the set S and indirectly in the strong sense $\widehat{P}$. Then there must be an alternative z in S in the presence of which x is directly revealed preferred to y. To put it more vividly, if one chooses chicken over steak, but chooses steak in a large menu including chicken, and also chooses steak over anything that is chosen over chicken, then steak must be chosen over chicken also in the presence of some element of the large menu. To illustrate further, note that the property is automatically verified for any set S of three alternatives, and in any set S of four alternatives it excludes the case in which the choice from S is never chosen in the presence of an alternative that dominates it. This is not an elegant axiom and we would not recommend it as a sufficient condition, but it is interesting as a necessary condition by providing a test of sequential rationalizability.

On the other hand, restricted Condorcet, considered before, is a rather attractive revealed preference property. We are now ready to prove that these two properties are observable implications of sequential rationalizability. This illustrates how sequential rationalizability limits the extent to which a choice function can be menu dependent.

Proposition 15 *Let γ be a sequentially rationalizable choice function. Then it satisfies (i) Menu and (ii) Restricted Condorcet.*

Proof. Let γ be sequentially rationalizable with $P_1, ..., P_K$ being the rationales. As usual, for any $(a,b) \in P_\gamma$, denote $i(a,b) = \min\{i : (a,b) \in P_i\}$.

(i) Menu. Let $(x,w) \in P_\gamma^{BInc}(S) \cap \widehat{P}_\gamma$. Note that for γ to be sequentially rationalizable clearly it must be $Up_\gamma(w,S) \neq \emptyset$. Now fix $z \in Up_\gamma(w,S)$. If $x = \gamma(wxz)$ we have nothing to prove. Otherwise, there can be two cases:

1. $w = \gamma(wxz)$. The sequential rationalizability of γ implies that $i(z,w) > i(w,x) \geq i(x,z)$ or $i(z,w) > i(w,x) \geq i(x,z)$ or $i(w,x) \geq i(z,w) > i(x,z)$. These are all the configurations ensuring that z is eliminated by x in wxz before z eliminates w. This however makes it impossible to rationalize $x = \gamma(S)$, unless there exists some alternative $y \in Up_\gamma(w,S)$ that eliminates w before it eliminates x. Since $y \in Up_\gamma(w,S)$ then by assumption $(x,y) \in P_\gamma$. So we have $(y,w) \in P_{i(y,w)}$ with $i(y,w) < i(w,x)$ and $i(x,y) \geq i(y,w)$. If $y = z$ this would generate a contradiction, so let $y \neq z$. Then the sequential application of the rationales to the set wxy yields $x = \gamma(wxy)$, and we are done.

2. $z = \gamma(wxz)$. The sequential rationalizability of γ implies that $i(x,z) > i(z,w) \geq i(w,x)$ or $i(x,z) \geq i(z,w) > i(w,x)$ or $i(z,w) \geq i(x,z) > i(w,x)$. These are all the configurations ensuring that x is eliminated by w in wxz before x eliminates z. As in case 1, this makes it impossible to rationalize $x = \gamma(S)$, unless there exists $y \in$

$Up_\gamma(w,S)$ such that $(y,w) \in P_{i(y,w)}$ with $i(y,w) < i(w,x)$ and $i(x,y) \geq i(y,w)$. The proof then concludes exactly as in case 1.

(ii) Restricted Condorcet. Let γ be sequentially rationalizable with $P_1, ..., P_K$ being the rationales.

Part (b) is proved in lemma 5. For part (a), let $x = \gamma(S)$. If $(x,y) \in P_\gamma$ for all $y \in S \backslash x$ we are done, so suppose that $Up_\gamma(x,S)$ is nonempty.

Since γ is rationalizable then $Lo_\gamma(x,S)$ is also nonempty. Suppose not. Then $(y,x) \in P_\gamma$ for all $y \in S \backslash x$ and therefore $(y,x) \in P_{i(x,y)}$, where $i(x,y)$ is defined as above. Therefore the successive application of the rationales to $S \backslash x$ must eliminate at each round exactly the same alternatives as it eliminates starting from S, namely $M_i(S) = M_i(S \backslash x) \cup x$ for all $i \in \{1, ..., K\}$. But this contradicts $x = \gamma(S) = M_K(S)$.

Clearly the rationalizability of γ also implies that, for all $y \in Up_\gamma(x,S)$, $Up_\gamma(y,S)$ is nonempty: otherwise any such y could never be eliminated by any rationale and it could not be that $x = \gamma(S)$.

Let R be the (possibly empty) subset of elements in $Up_\gamma(x,S)$ over which some element of $Lo_\gamma(x,S)$ is chosen in pairwise comparisons, that is:

$$R = \{y \in Up_\gamma(x,S) \,|Up_\gamma(y,S) \cap Lo_\gamma(x,S) \neq \varnothing\}$$

Let T be the (possibly empty) subset of elements in $Up_\gamma(x,S)$ to which some element of R is directly or indirectly revealed preferred in $Up_\gamma(x,S)$ via the base relation, that is:

$$T = \left\{y \in Up_\gamma(x,S) \,|\, (z,y) \in (P_\gamma | Up_\gamma(x,S))^t \text{ for some } z \in R\right\}$$

Finally let

$$Q = Up_\gamma(x,S) \setminus \{R \cup T\}$$

Suppose that Q is nonempty. Note that for all $y \in Q$ it must be that $(y,z) \in P_\gamma$ for all $z \in R \cup T$, since otherwise $(z,y) \in P_\gamma$ and therefore $y \in R \cup T$ (instead of $y \in Q$). Also obviously for all $y \in Q$ we have that $(y,z) \in P_\gamma$ for all $z \in Lo_\gamma(x,S)$, otherwise $y \in R$ (instead of $y \in Q$).

At any round of elimination, if an alternative in Q survives when the starting set is Q itself, so it must survive when the starting set is S, for there is no $z \in S \backslash Q$ such that $(z,y) \in P_\gamma$ for any $y \in Q$. Likewise, if an alternative in Q is eliminated at some round when the starting set is Q itself, so it must be eliminated when the starting set is S (by some other alternative also in Q). Therefore in particular $\gamma(Q)$ must survive all rounds of elimination when the rationales are applied to S, contradicting $x = \gamma(S)$ with γ rationalized by $P_1, ..., P_K$.

We can thus conclude that Q is empty. But then by the construction of the sets R, T, and $Lo_\gamma(x,S)$, for each $y \in R \cup T \cup Lo_\gamma(x,S)$ we have that $(x,y) \in (P_\gamma|S)^t$, as desired.■

Menu and Restricted Condorcet are not, however, sufficient for sequential rationalizability, as the following example demonstrates.

Example 16 $X = \{w,x,y,z\}$

$C_\gamma(w) = \{wx, wy, wxy, wzy\}$

$C_\gamma(x) = \{xy, xz, xyz, wxz, wxyz\}$

$C_\gamma(y) = \{yz\}$

$C_\gamma(z) = \{wz\}$

It is straightforward to check that γ satisfies Menu[6] and Restricted Condorcet.

In order to rationalize $x = \gamma(X)$ it must be that $i(z,w) \leq \min\{i(y,z), i(x,z)\}$. However this implies that $w = \gamma(wyz)$ cannot be rationalized.

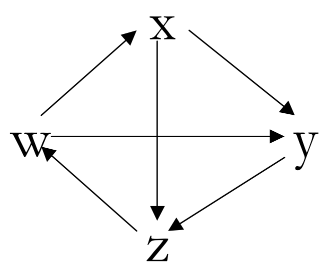

Figure 4: The base relation for example 16.

Finding a set of necessary and sufficient conditions for sequential rationalizability remains a non trivial open question.

6 Two applications of Rational Shortlist Methods

Among sequentially rationalizable choice functions, RSMs depart from conventional theories of rational choice in the smallest possible way. In section 3 we have shown how they rule out some well known procedures. In this section, conversely, we show how they can explain some remarkably regular departures from accepted principles of rational choice which go beyond the axioms of revealed preferences. We present two examples, one dealing with intertemporal choice and one with choice under risk.

[6]For instance $(x,w) \in P_\gamma(X) \cap \widehat{P}_\gamma \cap P^*_\gamma$ and as required by Menu $xwz \in C_\gamma(x)$.

6.1 Intertemporal choice

The standard model of choice over time is the exponential discounting model (EDM). It has been observed that actual choices in experimental settings violate consistently its predictions. The most notable violation is possibly preference reversal. Let P_γ refer to observed pairwise choices over date outcome pairs $(x,t) \in X \times T$, where X is a set of monetary outcomes and T is a set of dates. In this context, preference reversal at time t' is the shorthand for the following situation: $(x,t)\, P_\gamma\, (y,t')$ and $(y,t'+t'')\, P_\gamma\, (x,t+t'')$, where $t' \geq t$. This violates stationarity of time preferences, a premise on which the EDM model is constructed.

This choice pattern can be easily accounted for by interpreting γ as a RSM with rationales P_1 and P_2 defined as follows. For some function $u : X \times T \to \Re$ and number $\sigma > 0$, $(x,t)\, P_1\, (y,t')$ if and only if $u(x,t) > u(y,t') + \sigma$, and $(x,t)\, P_2\, (y,t')$ if and only if $u(y,t') \leq u(x,t) \leq u(y,t') + \sigma$, and either $x > y$, or $x = y$ and $t < t'$. This is compatible with preference reversal even with an exponential discounting type of u function. Let $x < y$, $t < t'$ and $u(x,t) = x\delta^t$ for $\delta \in (0,1)$. Suppose that $x\delta^t > y\delta^{t'} + \sigma$ so that (x,t) is chosen over (y,t'). Given σ, if t'' is sufficiently large it will be $x\delta^{t+t''} < y\delta^{t'+t''} + \sigma$, so that the two date outcome pairs are not comparable via P_1. That is, the decision maker looks first at discounted value, and chooses one alternative over the other if it exceeds the discounted value of the latter by an amount of at least σ. Otherwise he looks first at the outcome dimension and if this is not decisive at the time dimension. The application of P_2 yields the choice of $(y,t'+t'')$ over $(x,t+t'')$, thus 'reversing the (revealed) preference'.

Obviously, P_γ could also be sequentially rationalized by using three rationales, where the outcome and time dimension comparisons are used in two separate P_i.

The same model can explain cyclical intertemporal choices and other 'anomalies' (see Manzini and Mariotti [14] and bibliography therein).

6.2 Choice under risk

A model similar to that of the previous section can be employed to deal with some departures of (experimentally) observed choice from Expected Utility (EU), the standard tool of analysis of choice under risk. Consider elementary lotteries of the form $(x,p) \in X \times [0,1]$ where X is a set of monetary prizes and p is the probability with which prize x is obtained while $1-p$ is the probability of obtaining zero. In this context the 'common ratio effect' refers to the following situation: $(x,p)\, P_\gamma\, (y,q)$ and $(y,rq)\, P_\gamma\, (x,rp)$, for some $r \in (0,1)$. This violates the Independence Axiom for choice over lotteries.

Define rationales P_1 and P_2 as follows. Let $U : X \times [0,1] \to \Re$ be an expected utility function defined by $U(x,p) = pu(x)$, where $u : X \to \Re$ is a concave function, and let

$\sigma > 0$. Let $(x, p)\, P_1\,(y, q)$ if and only if $U(x, p) > U(y, q) + \sigma$. Let $(x, p)\, P_2\,(y, q)$ if and only if $U(y, q) \leq U(x, p) \leq U(y, q) + \sigma$, and either $x > y$ or $x = y$ and $p > q$. We now provide a famous numerical example (see Kahneman and Tversky [9]) of actual choices that can be explained in the context of RSM.

Consider the gambles $g_1 = (4000, 0.8)$, $g_2 = (3000, 1)$, $g_3 = (4000, 0.2)$ and $g_4 = (3000, 0.25)$. Note that g_3 and g_4 are the same as g_1 and g_2, respectively, with the probability of the positive prize reduced by a factor $r = 0.25$. In experiments it is normally found that a significant majority of choosers picks g_2 over g_1 and g_3 over g_4, violating independence and EU. This pattern of choice requires that

$$
\begin{aligned}
&u(3000) > (0.8)\,u(4000) + \sigma \\
&(0.25)\,u(3000) \leq (0.2)\,u(4000) + \sigma \\
&(0.2)\,u(4000) \leq (0.25)\,u(3000) + \sigma
\end{aligned}
$$

The first line ensures that $g_2 P_1 g_1$, while the last two inequalities imply that $g_3 P_2 g_4$. The inequalities are compatible because there exists a positive constant σ and a concave u such that $\sigma < u(3000) - (0.8)\,u(4000)$ and $\sigma \geq |(0.2)\,u(4000) - (0.25)\,u(3000)|$. For instance, take $u(x) = \ln(x + 1)$ and $\sigma \in [0.343, 1.371)$. This example is taken from [15] to which we refer for further details. Interestingly, in his work on similarity Rubinstein [20] also suggests a (partially unspecified) sequential procedure to evaluate elementary lotteries. Since there is a formal connection between a similarity relation and a semiorder, the approach proposed here is in the same spirit as Rubinstein [20]. However note that crucially he focuses on similarities only on each dimension.

7 Concluding Remarks

Both economists and psychologists have tended to emphasize the contrast between 'hyper-rational' decision making and choice based on rules of thumb. An economist may feel that the psychology way of proceeding is too loose, in that by using appropriate sequences of rationales everything, or at least too much, can be explained. Indeed, the almost universal use of utility functions in economic models confirms that uniqueness of the rationale (and much more) is implicitly assumed. On the other hand, the existence of an overarching preference relation as hypothesized by economists is very stringent, creating a wide gap between what is rationalizable according to economist and what is normally considered 'reasonable' behavior in practice.

We have proposed an economic approach to the type of decision making procedures which are usually promoted by psychologists. For example Gigerenzer and Todd [7] in their work on 'fast and frugal' heuristics observe that "One way to select a single option from multiple alternatives is to follow the simple principle of elimination: successive cues are used to eliminate more and more alternatives and thereby reduce the set of remaining options, until a single option can be decided upon.". In general, such heuristics focus on the simplicity of cues used to narrow down possible candidates for choice. Simplicity is an essential virtue in a world in which time is pressing. An overarching preference relation - let alone a utility function - is not a cognitively simple object, and as a consequence these authors stress the difference from heuristics based reasoning and the 'unlimited demonic or supernatural reasoning' relied upon in economics. Yet in this paper we have shown that the standard tools and concepts of revealed preference theory can be used to formalize such heuristics.

Our way of incorporating bounded rationality is to translate the psychological notion of 'cues' into a set of *not necessarily complete* binary relations. Rationality for us is the *consistent* application of a sequence of rationales. The order in which they are applied may be hardwired and may depend on the specific context and on the type of decision maker[7], but it should be the same in a relevant class of decision problems. Each single rationale in itself needs not exhibit any other strong property, such as transitivity.

Sen [21] has forcefully argued that there are many reasons why preferences may be incomplete[8]. As his main objective is to show how the act of *maximization* (as opposed to optimization) of a binary relation is related to the act of choice, he does not force choice to be single valued. Eliaz and Ok [4] also look at choice correspondences that can be explained by the maximization of a (single) incomplete preference relation. In addition, they provide experimental evidence for a weakening of WARP which characterizes such correspondences. Here we are instead interested in 'inescapable' choices, namely in how a *specific,* single valued, act of choice is arrived at, and this is why we consider the sequential application of incomplete binary relations.

The usefulness of elimination heuristics in practical decision making is self-evident[9] and widely spread in fields such as clinical medicine. In this perspective the *sequentiality* in the application of rationales, which lies at the core of our analysis, is an appealing feature of our rationalization results. This is very different from rationalization by multi-

[7]For example, in order to 'choose' whether to stay or flee in the presence of a bird, a rabbit may use as its first rationale the fact that bird is gliding, which would identify a predator. Conversely, a human decision maker may well look first at size or shape in order to recognize the bird.

[8]Beside those cited in the text, Danan [3], Mandler [12], Manzini and Mariotti [15], Masatlioglu and Ok[16] are all contributions that have, from various perspectives, advocated the use of incomplete preference relations. Danan and Masatlioglu and Ok in particular have looked at the rationalization issue.

[9]As put very effectively by Gigerenzer and Todd [7] "If we can decide quickly and with few cues whether an approaching person or bear is interested in fighting, playing, or courting, we will have more time to prepare and act accordingly (though in the case of the bear all three intentions may be equally unappealing)".

ple rationales, known in the literature as *pseudo-rationalization*[10], whereby all rationales are applied simultaneously to each set. Our approach is also different from the recent contribution by Kalai, Spiegler and Rubinstein [10]. They use multiple rationales to explain choices, but each rationale is applied to a subset of the domain of choice. This results in all choices being rationalizable and the focus becomes that of 'counting' the minimum number of rationales necessary to explain choices. Finally we should mention the work by Ok [19] who characterizes the choice correspondences satisfying Independence of Irrelevant Alternatives by means of a two-stage procedure.

References

[1] Aizerman, A. (1985) "New Problems in the General Choice Theory", *Social Choice and Welfare*, 2, 235-282.

[2] Baigent, N. and W. Gaertner (1996) "Never Choose the Uniquely Largest: A Characterization", *Economic Theory*, 8, 239-249.

[3] Danan, E. (2003) "A Behavioral Model of Individual Welfare" mimeo EUREQua, Universite' de Paris I', mimeo.

[4] Eliaz, K. and E. Ok (2004) "Indifference or Indecisiveness? Choice-Theoretic Foundations of Incomplete Preferences", mimeo, NYU.

[5] Gaertner, W. and Y. Xu (1999) "On Rationalizability of Choice Functions: A Characterization of the Median", *Social Choice and Welfare*, 16, 629-638.

[6] Gaertner, W. and Y. Xu (1999) "On The Structure of Choice Under Different External References", *Economic Theory*, 14, 609-620.

[7] Gigerenzer, G., P. Todd, and the ABC Research Group (1999) *Simple Heuristics That Make Us Smart.* New York: Oxford University Press..

[8] Gigerenzer, G., and P. Todd (1999) "Fast and Frugal Heuristics: The Adaptive Toolbox", in G. Gigerenzer, P Todd, and the ABC Research Group, *Simple heuristics that make us smart.* New York: Oxford University Press.

[9] Kahneman, D. and A. Tversky (1979) "Prospect Theory: an Analysis of Decision Under Risk", *Econometrica*, 47, 263-291.

[10] Kalai, G., A. Rubinstein and R. Spiegler (2002) "Rationalizing Choice Functions by Multiple Rationales", *Econometrica*, 70, 2481-2488.

[10] See Aizerman [1] and Nehring [18].

[11] Luce, D. and H. Raiffa (1957) *Games and Decisions: Introduction and Critical Survey*, New York: Wiley.

[12] Mandler, M. (2001) "Incomplete Preferences and Rational Intransitive Choice", mimeo, Royal Holloway, University of London.

[13] Manzini, P. and M. Mariotti (2002) "A Vague Theory of Choice Over Time", Working paper EconWPA ewp-game/0304003,

[14] Manzini, P. and M. Mariotti (2003a) "A Theory of Vague Expected Utility", Working paper EconWPA ewp-game/0203004.

[15] Manzini, P. and M. Mariotti (2003b) "How vague can one be? Rational preferences without completeness or transitivity", Working paper EconWPA ewp-game/0312006.

[16] Masatlioglu, Y. and E. Ok (forthcoming) "Rational Choice with a Status-Quo Bias", *Journal of Economic Theory.*

[17] Moulin, H. (1985) "Choice Functions Over a Finite Set: A Summary" *Social Choice and Welfare*, 2, 147-160.

[18] Nehring, K. (1997) "Rational Choice and Revealed Preference without Binariness", *Social Choice and Welfare, 14, 403-425.*

[19] Ok, E. (2004) "Independence of Irrelevant Alternatives and Individual Choice ", mimeo, NYU.

[20] Rubinstein, A. (1988) "Similarity and Decision-Making Under Risk", *Journal of Economic Theory*, 46, 145-153.

[21] Sen, A. (1997) "Maximization and the Act of Choice", *Econometrica*, 65, 745-779.

[22] Suzumura, K. (1983) *Rational Choice, Collective Decisions, and Social Welfare*, Cambridge University Press, Cambridge U.K.

[23] Tversky, A. (1972) "Elimination By Aspects: A Theory of Choice", *Psychological Review*, 79, 281-299.