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ABSTRACT

Samuelson's Contributions to Population Theory and Overlapping Generations in Economics*

Paul Samuelson made a series of important contributions to population theory for humans and other species, evolutionary theory, and the theory of age structured life cycles in economic equilibrium and growth. The work is highly abstract but much of it was intended to illuminate issues of compelling policy importance, such as declining fertility and population aging. While his work in population economics has been very influential, his work in population and evolution appears to have been largely overlooked, perhaps because he seldom published in demographic journals or went to population meetings. Here I discuss his many contributions in all these areas, but give particular attention to demographic aspects of his famous work on overlapping generation models, social security systems, and population growth.

JEL Classification: J11, J18, Q57, H55, D64

Keywords: Samuelson, overlapping generations, social security, population growth rate, reproductive value, population cycles, intergenerational transfers, predator-prey, altruism

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Paul Samuelson's death in 2009 was followed by an explosion of biographical articles extolling his many seminal contributions. Many of these articles mention that Samuelson was "born" as an economist when, as an undergraduate at the University of Chicago, he attended a lecture about Thomas Malthus. Malthus, of course, propounded a seminal theory of how population and the economy interacted. But I was disappointed to find that none of the many articles I read made any other mention of population in relation to Samuelson's work, except for Dixit (2012). This strikes me as odd, because as an economic demographer myself, I view Samuelson's contributions to demography and its interface with economics as deep and important.

The work that might be claimed for economic demography, such as his famous paper on the consumption loan economy with overlapping generations, is very well known and enormously influential. But his other work on population, some of it also related to biological evolution, is barely known or remembered at all, to the detriment of the field. Who now knows that Samuelson published two papers on the two-sex problem in demography? Or two papers on predator-prey models? A paper on the reversibility of time in population processes? Five papers generalizing and critiquing Fisher's concept of reproductive value, a concept that plays a key role in evolutionary theory? Analyses of evolutionary theories of altruism and kin selection? I think this work has remained largely overlooked because he very rarely published in population journals (I know of only one instance). His published work on population is scattered among biological and general science journals for the most part. His demographic work has never been collected in a single volume targeted at a demographic readership. So far as I know, Samuelson never went to population meetings or sought to publicize this work in the demographic community. Furthermore, the analysis in his papers is highly mathematical, and much is too difficult for most demographers, even those who, like myself, think of themselves as mathematical types. It is clear from comments in his papers that he often discussed his work with the leading mathematical demographer Nathan Keyfitz. In the 1970s, when he did much of this work, he had a grant from the National Institute of Child Health and Development (NICHD) to pursue these issues. To attempt to draw together some of these various lines of Samuelson's work here is an opportunity, a challenge, an education, and a pleasure.

Much of Samuelson's work in population was about population dynamics, that is change over time. He particularly focused on the constraints on change that were often omitted in analyses by demographers, such as natural resource limits. For example, in his *Foundations of Economic Analysis* he develops a formalization of the basic Malthusian model of population growth and equilibrium, noting that the equilibrium was stable under Malthus' assumptions. He also develops a simple model of the optimal size for a population that first experiences increasing returns and then decreasing returns to labor. He makes the useful point that the optimal population size could not be maintained as the equilibrium in a Malthusian type system, or we might now say under population homeostasis. The reason is that the equilibrium is only stable on the up side; on the down side, it is unstable. A positive shock to population size would reduce per capita income so population would shrink back toward the optimum size. But a negative shock would also reduce per capita income, so the population would shrink further, and further reduce per capita income and population size in this region of increasing returns to labor, rather than converging back up to the optimum size and income.

When population age distribution is integrated into the theory, the dynamic issues become more complicated, and this also interested Samuelson. Most of his work on population occurred in the 1970s and early 1980s when the "New Home Economics" approach to household behavior, and particularly fertility, was prominent through the work of Becker (1960, 1981, and with Lewis, 1973) and Willis (1974) among others. Samuelson was not a fan of this line of work, which he contrasted unfavorably to the more socio-economic-demographic approach of Richard Easterlin (1968): "The Easterlin theory is all the more valuable for its scarcity among economic theories, standing out in welcome relief from the rather sterile verbalizations by which economists have tended to describe fertility decisions in terms of the jargon of indifference curves, thereby tending to intimidate non-economists who have not misspent their youth in mastering the intricacies of modern utility theory." (Samuelson, 1976b:244). Samuelson's demographic work took a rather macro approach, as opposed to the microeconomic theories he described as sterile. This is also true of Easterlin's theory, which emphasized aggregate level economic feedbacks within the context of a fertility theory in which the income of each generation relative to that of its parents drove the generation's marriage, fertility, and labor force participation. Malthus and other Classical economists claimed that the long run supply of labor was infinite at a wage equal to the subsistence level, while suggesting that this subsistence level depended on workers' changing notions of what consumption was necessary. Samuelson (1985) viewed this approach as vacuous, while Easterlin's idea of relative income was well-defined and operationalized by comparing income levels of children and parents.

Easterlin noted that the small generations born during the Great Depression of the 1930s in the US grew up to experience high wages and easy labor market entry as young adults, marrying early and having high fertility in the 1950s and 1960s. This small generation of parents produced a large generation of children, giving rise to the Baby Boom. Easterlin predicted that as this large generation aged, it would experience relatively lower wages, slower promotion, later marriage and lower fertility – all of which came to pass in the Baby Bust years of the 1970s. Samuelson (1976b:244) says "Thus the Easterlin hypothesis can explain fertility waves not unlike those actually experienced in the United States during the last 40 years." Samuelson proceeds to develop a simple model of the Easterlin theory and analyzes its dynamic behavior, showing that depending on sensitivities, it may have an unstable equilibrium, away from which it moves to a self-generating limit cycle with oscillations every other generation, much as Easterlin described (1968).[1]

A different kind of constraint arises in the so-called "two sex problem" in demography. Most demographic models and analyses are based on the single sex population. For example, population projections are virtually always done for the female population, based on measured and projected female fertility and mortality rates. Once the female population has been projected, including future female births, then the sex ratio at birth is used to derive the future numbers of male births from the female ones. Then future male mortality can be used to complete the projection of the male population by age. This is called a "female dominant" projection. Just as actual real world projections almost always ignore the possibility that natural resource constraints will affect future population growth, so they also ignore constraints arising from the requirement that male and female fertilities and growth rates are interlinked in a nonlinear way. This is a classical problem in mathematical demography, and the

[1] I myself had published a very similar analysis in Lee (1974), which Samuelson generously acknowledged but learned about too late to take into account in his own analysis, for the most part.

contributions to it by Yellin and Samuelson (1974, 1977) draw on that earlier work. Pollak (1986, 1990) made seminal contributions to this problem since then. Yellin and Samuelson (1977) proved a theorem for the general case of an age structured population with asymmetric birth functions (e.g. young females have births with both younger and older men, but younger men are unlikely to have births with older women) and non-diagonal birth functions (so males and females of different ages have births). They show that for arbitrary initial age distributions for each sex, the female dominant and male dominant single sex growth rates need not bracket the true non-linearly generated rate, but they also show that it is always possible to find initial age distributions sufficiently close to the stable case such that the true rate is bracketed by the single sex rates. Well, this is good to know. However, the demographer in me is disappointed that they did not discuss whether all this matters from a practical point of view. I would have liked to see some realistic simulations over a 50 to 100 year time horizon to learn whether the two-sex nonlinearity is a practical problem for projections.

Samuelson continued on this path of careful mathematical analysis of central issues in population theory in greater generality, in particular in the presence of two-sex dynamics and environmental constraints, with five papers between 1977 and 1980 on Fisher's "reproductive value". This concept, formalized by Fisher in 2030, quantifies the expected number of female descendants of a given female at a given age, discounted at the steady state population growth rate. It is therefore closely related to success in achieving the transmission of genes into future generations, which drives evolution by natural selection. As explained in a recent article in *Genetics* "In the long term, alleles that increase the reproductive value will be the ones that increase, and traits will evolve that tend to maximize an individual's reproductive value." (Barton and Etheridge, 2011). Samuelson's first two papers on this topic (1977a,b) considered reproductive value in populations that were sufficiently sparse that natural resource constraints would not limit their growth in the medium term, so linear models of population growth could be used, and birth functions (depending on numbers of both males and females) are linear homogeneous: doubling the numbers of males and females would double the numbers of births. For this case he then analyzed biparental reproduction, drawing on his earlier work with Yellin on the two-sex problem. He was able to show that important properties of Fisher's reproductive value remained valid in this case of sparse population relative to natural resources. However, he also found that in the more realistic case of a population constrained by the carrying capacity of its environment, the reproductive value would be zero for females at every age, because varying the size of age elements of the initial population would have no effect on long run equilibrium population, which is determined only by environmental carrying capacity. In Samuelson (1978a) he shows that finding the reproductive values in the two-sex case would involve an infinite number of calculations, meaning that the concept can no longer be used as Fisher (1930:25) had hoped, in relation to his Fundamental Theorem of Natural Selection. In his next paper on this topic, Samuelson (1978b) considers how the situation might be at least partially saved by considering approximations. First, he suggests that even if the long run population dynamics are constrained by the environment, nonetheless at very low population sizes the dynamics will be close to linear, and the reproductive value calculation can be of some use for demography and genetics. Second, he suggests that a somewhat modified calculation can provide useful information in the neighborhood of the equilibrium size determined by the carrying capacity.

Samuelson did additional work at the interface of biology and demography. One topic, the Lotka-Volterra predator-prey model, is analyzed in two articles. In the predator prey model the growth rate of predators is positively related to the abuncance of prey and negatively related to the number of predators, while the growth rate of prey is negatively related to the abundance of predators. Samuelson sometimes interprets predators as capitalists and laborers as prey. This model generates periodic cycles in the sizes of both populations. In 1971 he generalizes the math of the original model to many interacting species and to other functional forms. He then goes on to develop more realistic models in which there are environmental resource constraints via diminishing returns. “Ecological equilibrium without the law of diminishing returns is like Hamlet without the Prince” (1971:982). In these models, cycles are no longer self-exciting, instead spiraling down to equilibrium. But cycles can be sustained by continuing external shocks such as weather. An article in 1974 further generalizes and simplifies the mathematics.

In 1983 Samuelson took on a new problem: the evolution of altruism through kin selection, a centrally important topic in evolutionary theory since the second half of the 20th century, with seminal contributions by Hamilton on altruism and kin selection, by Wilson on sociobiology, and by Trivers on other applications of kin selection. Samuelson, in his famous 1958 essay, wrote: “The economics of collusion provides an important field of study for the theorist. Such collusions can be important elements of strength in the struggle for existence. … But culture in which altruism abounds - because men do not think to behave like atomistic competitors or because men have by custom and law entered into binding social contracts - may have great survival and expansion powers.” (p.481) It is a small step from acknowledging these powers to asking why they might not evolve through some sort of natural selection, and that is the topic of his 1983 paper. I think it is fair to say that on this topic Samuelson introduced more powerful, comprehensive and general mathematical treatments of the earlier theories, and cleared up some points and controversies, but did not fundamentally advance the topic in the way he did so many others.

Some other contributions by Samuelson were more purely demographic, such as his valuable work on the time symmetry and asymmetry of forward and backward dynamics in population (1976c). It is well-established that going forward in time, populations tend to forget their past, in the sense that from different initial age distributions they will converge to the same final age distribution if subjected to the same trajectories of fertility and mortality. But this feature also makes the attempt to infer earlier age distributions by going backward (reversing the projection dynamics) impossible or exceedingly data-demanding, and this difficulty is compounded by the problem of choosing how many old people to move into and out of the open oldest age interval as we go backward. This may sound like an obscure academic point, but it is actually of practical importance in attempts to reconstruct the demographic past, and it has sometimes led to confusion and controversy. An example is a method called “inverse projection” in which, going forward from a given initial population, series of births and deaths (derived from records of baptisms and burials) are used to estimate future population sizes and age distributions, while a method called “back projection” sought (unsuccessfully, in my view) to perform a similar procedure backward in time, from a given more recent population age distribution and again using series of births and deaths (Lee, 1985, 1993; Vaupel et al, 2004; Wrigley and Schofield, 1981). Samuelson (1976c) made the point that mathematically, this has nothing to do with ergodicity of population age structures, since convergence is never complete and if one has an exact stable population age distribution at some time then, with constant rates in the past, it must have always been stable. Instead, the problem comes from the singularity of the Leslie Matrix, derived from the problem of the open age interval at the top. But I should add here that practically, given the limits to accuracy of demographic data, particularly historical data, the problem of age distribution convergence is indeed key in this context, and unknown levels of net migration during the study period add a further layer of uncertainty.

The work discussed up to this point should be of great interest to demographers and evolutionary biologists, but now it is time to turn to Samuelson's work at the interface of demography and economics. This seminal work originates at an earlier date, with his famous Consumption Loan paper published in 1958, and then continues in the 1970s with papers on optimal population growth rates (1975b) and optimal Social Security programs (1975a), and with comments and rejoinders to this work. Doubtless there are other relevant contributions by Samuelson, but I will focus on these issues and papers.

Consumption-Loan Model with Overlapping Generations

Robinson Crusoe was on his own as he moved through his lifecycle. Most people, though, live in populations that include individuals at every age and every stage of the economic life cycle. Such individuals interact through economic markets but they also give and receive support through social relationships such as parent-child, elder and adult offspring, tax payer and public beneficiary, and giver/receiver of bequests, which I will refer to as "transfers". One reason to use "overlapping" generation models to analyze general equilibrium outcomes is to take into account the relative numbers of actors at different ages/stages of their life cycles. As relative numbers of people at different ages change, this will affect their interactions through markets and through social relations, and cause adjustments in prices (e.g. wages and interest rates), social transfers (e.g. spending per child, public pensions, taxes) and life cycle behavior (e.g. saving, age at retirement). Robinson Crusoe could save, accumulate capital, and dissave, and in these ways reallocate his income to some degree across his life cycle, for example by building a dwelling that yielded services over the long run, or storing food for later consumption. Most people, however, have a much richer menu of ways to shift income across their life cycles. Like Crusoe, they can save, invest, and dissave. But in addition they can borrow and lend to others, and they can give and receive public and private transfers.

Samuelson (1958) set up his analysis to rule out all durable goods including capital. This meant that on the one hand, accumulation of consumption goods could not be used to shift income to later in the life cycle, and on the other, construction of productive assets like a home or machine was not possible. This leaves two ways to reallocate income across the life cycle: transfers (either public or private) or credit markets. Credit markets could be used only for "consumption loans", not to fund capital investment, since capital does not exist.

Samuelson (1958:468) develops his model with three adult ages -- two ages of workers and one of retirees. "Let us assume that men enter the labor market at about the age of twenty. They work for forty-five years or so and then live for fifteen years in retirement. (As children they are part of their parents' consumptions, and we take no note of them.)" He later comments (1958:475): "By introducing overlap between workers of different ages, the three-period model is essentially equivalent to a general n-period model or to the continuous-time model of real life." The model with three age groups is amenable to analytic solutions and is quite intuitive, and since children don't make their own consumption decisions it can make sense to view them as a part of their parents' consumption.

Nonetheless, it may be of interest both theoretically and empirically to consider the "continuous-time model of real life" with realistic demography in which people can be any age, are constantly at risk of death, and live part of their lives as children (Lee, 1994; Bommier and Lee, 2003). We can give them more realistic economic life cycles in which productivity and labor effort vary across age, and even children and the elderly work to some extent, albeit less productively. Children do not participate in credit markets, but they do consume by virtue of receiving transfers from their parents or the public sector.

Consider a world with no capital, output is produced solely from labor, and there are no durable goods--Samuelson's consumption loan world. People produce an amount y(x) at age x, which will be taken as given. They choose a lifetime consumption trajectory c(x), presumably by seeking to maximize lifetime utility, but the details of the origin of c(x) need not concern us here. However, given the earnings age schedule, consumption is constrained by the life cycle budget constraint, which for interest rate r, is given by:

(0.1) $$\int_0^{\omega} e^{-rx} p(x)\left[y(x)-c(x)\right]dx = 0$$

There is also a social budget constraint that must be satisfied, and it also limits the consumption possibilities. Since there are no durable goods, and therefore it is impossible to borrow or loan from Nature (in Samuelson's words), total consumption in each period must be no more than total output; since it would be inefficient to leave excess output unconsumed, output and consumption must be equal in any efficient program. The social budget constraint depends in part on the population age distribution, and we will assume it is in steady state, or what demographers call "stable". Let p(x) be the probability of surviving from birth to age x, and let p(x) be 0 for $x \geq \omega$. The age specific birth rates, together with the survival schedule, will generate a stable growth rate n and a corresponding population age distribution. In the stable population the number of births grows exponentially at rate n, so at time t the number of people age x will be $B(0)e^{-n(t-x)}p(x)$. The total population is the integral of this over x, and *B(0)* and divided by the total population is the crude birth rate, b. From this it follows that the proportion of the population at age x is just $be^{-n(t-x)}p(x)$, and these proportions will, of course, integrate to 1.0.

The social budget constraint is then:

(0.2) $$b\int_0^{\omega} e^{-nx} p(x)\left[y(x)-c(x)\right]dx = 0$$

We can immediately note that a rate of interest equal to the population growth rate, r=n, satisfies both the individual and the social budget constraints. This is Samuelson's biological interest rate result.

If the c(x) consumption path has been chosen to maximize lifetime expected utility given interest rate r, then we can also note that the biological interest rate program is optimal, in the sense that it maximizes steady state utility subject to the social budget constraint. It is therefore Pareto optimal, as in Samuelson's model with three age groups.

The question now arises: is the Pareto optimal outcome with r=n attainable as a competitive equilibrium? To address this question, we begin by noting that there is one additional constraint which holds for credit markets in the aggregate. If the economy is closed and that there is no government debt or credit then for every dollar of credit in the economy there must also be a corresponding dollar of debt, so that the aggregate net credit balance is identically 0. We can calculate this credit balance as follows. Consider an individual at age x. Looking to the future, and taking into account the probability of surviving from x to each future age a, this individual's net credit balance (the amount of credit the individual would need to hold at age x in order to balance the future budget, given future earnings and mortality) is the present value at age x of all future differences between what is consumed and what is earned, which Samuelson called "'net' or excess demands", negative excess demands, or negative "net saving" (1958:470).

(0.3) $$\int_x^{\omega} e^{-r(a-x)} \left[p(a)/p(x) \right] \left[c(a) - y(a) \right] da$$

The aggregate credit balance is found by integrating this across all ages, weighting by the number of people at each age, and dividing by the total population to get a per capita value (or just weighting by the proportional age distribution derived above):

(0.4) $$W = b\int_0^{\omega} e^{-nx} \int_x^{\omega} e^{-r(a-x)} p(a) \left[y(a) - c(a) \right] da\, dx$$

Now consider the special case of the biological interest rate at which the optimal life cycle consumption occurs, r=n. After some manipulation as described in Bommier and Lee (2003) and Lee (1994), building on Gale (1973) and Willis (1988), we find the elegant result (due to Willis, 1988):

(0.5) $$W = c\left(A_c - A_y \right).$$

In this expression, c is aggregate per capita consumption, A_c is the average age of consuming in the population (the age at which the average dollar of consumption takes place in the population), and A_{yl} is the average age of earning in the population. This tells us that if the average age of consuming is greater than the average age of earning, then the population as a whole will have to hold a positive amount of wealth W, because the average unit of output will be held for $A_c - A_y$ years before it is consumed. Here is the definition of the average age of consuming, for example:

(0.6) $$A_c = \int_0^{\omega} x e^{-nx} p(x) c(x)\, dx \Big/ \int_0^{\omega} e^{-nx} p(x) c(x)\, dx$$

Evidently this average age depends both on the shape of consumption across the lifecycle, c(x), and on the age distribution of the population by which it is weighted.

Returning to our question of whether credit markets can support the biological interest rate, note that in a closed economy, for every dollar of debt there is a corresponding dollar of credit, so W, aggregate credit, must be identically equal to zero. While individuals can smooth consumption across age by borrowing and lending, for society as a whole this is not an option. The credit market can support only reallocations of income that are symmetric with age. Therefore, it cannot support the optimal age pattern of consumption that corresponds to the biological interest rate n, except in the probability zero case where the corresponding average ages happen to be equal. In particular, it could not support the age patterns that Samuelson assumes, with work when young and retirement when old, which is highly asymmetric. Indeed, in Samuelson's example, in a stationary population with equal consumption by young workers, old workers, and the retired, and equal earning by the two worker groups, if ages are 1, 2 and 3 then the average age of consuming is 2 and of working is 1.5, so the difference is .5 time units, or ten years in Samuelson's example of 20-year age groups (1958:473).

In fact, as Samuelson (1958:478) reports, the credit market outcome for his three age group model with a logarithmic utility function would be a negative interest rate of .70 per 20-year period, or about -2% per year. At this negative interest rate, young workers choose to borrow from older workers rather than lend, and then they repay these older workers once they retire. The result is a very unappealing pattern of lifecycle consumption, with very high consumption when young and very low when old, yielding much lower lifetime utility than the optimal pattern under the biological interest rate of zero for a stationary population. Samuelson (1958:479) comments "If each man insists on a quid pro quo, we apparently continue until the end of time, with each worse off than in the social optimum, biological interest case." He reaches this conclusion by starting with Adam at the beginning of time, and making sure that every borrowing is offset by a loan, but the argument above based on the aggregate credit balance seems simpler. (If Samuelson were here, he might demolish the argument in one withering sentence, however.)

How, then, can the biological interest rate outcome be achieved? Samuelson (1958:479-480) continues "Yet how easy it is by a simple change in the rules of the game to get to the optimum. Let mankind enter into a Hobbes-Rousseau social contract in which the young are assured of their retirement subsistence if they will today support the aged, such support to be guaranteed by a draft on the yet-unborn. Then the social optimum can be achieved within one lifetime."

In other words, nonmarket social transfers can get us to the optimum. Here is how. Let the gap between consumption and earnings at any age be filled by a transfer $\tau^+(x)$ received from other ages at which earnings exceed consumption, and which make an average transfer $\tau^-(x)$, with the net transfer received at each age given by $\tau(x) = \tau^+(x) - \tau^-(x)$. Then the aggregate transfer balance, T, can be calculated in the same way as the aggregate credit balance:

(0.7) $$T = \tau^+\left(A_{\tau^+} - A_{\tau^-}\right)$$

Like borrowing and lending, the flow of transfers given and received must exactly cancel at any moment. But unlike credit, the aggregate transfer balance, which we might call "transfer wealth", can take any value, positive or negative. For example, in a public pension system, workers pay taxes and retirees receive benefits, and the transfer flow is solely from young to old and highly asymmetric, with a large positive value for T. By contrast, in a society where the elderly continue to work and all transfers go to children, T would have a large negative value, as is true in hunter-gatherer societies. The point is that such non-market transfers, whether public or private, can support the optimal age pattern of consumption, through the "social contract" that Samuelson referred to above. They do this through the device of committing those not yet alive, the unborn but to-be-born, to either make transfers in the future or in some cases to receive them. In addition, the "social contrivance of money", as Samuelson puts it, can bring about the same pattern of intergenerational transfers through induced inflation, in the way he describes in his paper – but in practice I find this is an unappealing interpretation of his results.

To give all this empirical context, Figure 1 uses arrows to plot average ages of earning labor income and consuming in many countries all around the world[2], and also in two contemporary hunter-gatherer

[2] These data come from the National Transfer Accounts project (see Lee and Mason et al, 2011, and United Nations Population Division, 2013), which was inspired in part by Samuelson (1958). Using a mathematical framework

groups. We see that the actual direction of income reallocation is downward, from older to younger, in most low and middle income countries, but has reversed in a half dozen rich industrial nations. These patterns are partly due to the older population age distributions of most rich industrial nations, and the younger age distributions in most non-rich other countries, but that is far from the whole story. In rich nations labor supply ends at younger ages, facilitated or incentivized by public and private pension systems, and in many of these nations consumption at older ages has grown more rapidly than at younger adult ages, tilting the consumption age profile upward, due to the rising costs of publicly provided health care.

Adding Capital – Now What is Optimal?

Samuelson developed his consumption-loan model the way he did, with no capital and no durable good, because he wanted to show that these were not necessary underpinnings of the interest rate, and therefore not its fundamental source. But durable goods and productive capital can readily be added to the model, as Diamond (1965) did. In the Solow model, for a given saving rate, the slower the population growth the better, because that leads to more capital per worker and higher productivity and per capita income. But in the consumption loan model *faster* population growth is beneficial because it means more workers per retiree, so everyone can be better off. As Samuelson (1975b:531) put it: "In the usual version of that growth model, the slower the rate of exponential growth the higher can be the level of steady-state consumption; on the other hand, in a life-cycle pure-consumption model of the 1958 Samuelson type, which permits no trades with nature, the faster the population growth the better, since more children means better support for retired parents." It seems plausible that there would be an intermediate level of the population growth rate at which the tradeoff between capital dilution and dependency ratio maximizes economic wellbeing, at least in this rarefied world in which natural resource constraints are ignored. And this was the argument and analysis in Samuelson's 1975b paper on "The Optimum Growth Rate for Population".

For a given population growth rate n, we can calculate the optimal steady state saving rate, s*, per capita consumption c* and capital labor ratio, k*. Now we can ask how the optimal level of consumption, c*, is affected by the population growth rate n. Samuelson gives the answer:

(0.8) $$dc^*/dn = -k^*$$

If δ is the rate of depreciation, then n= $-\delta$ is the most rapid rate of population decline consistent with a steady state, so that is the optimal rate of population growth that maximizes steady state consumption.

Samuelson then derives the "two part golden rule", similar to the standard one for the Solow model, but with workers and retirees who arrange life cycle consumption as guided by the biological interest rate of g. Now Samuelson can ask: what is the goldenest golden rule which gives maximum lifetime utility when we vary both saving rates and the population growth rate? And he provides an answer (1975b:534). He then proves his famous "serendipity theorem". He notes that in general, for a given population growth

much like that presented in this chapter, the project estimates age profiles of consumption, labor income, saving, asset income, and public and private transfers by age for about 80 countries in the world. There are research teams 55 countries in Europe, Asia, Africa, Latin America, North America and Oceania making these estimates according to a strict methodology set out in a manual published by the United Nations (2013). The project website, where much of this data can be found, is ntacounts.org.

rate n, the private saving induced by lifecycle saving will not correspond to the rate needed to achieve the golden rule. To achieve the golden rule would require intervention. For example, a Pay-As-You-Go public pension system would substitute for some private saving, which would move the economy toward the golden rule if the capital labor ratio were inefficiently high. Similarly, if the capital labor ratio were too low, he suggests that a funded compulsory public pension system in which contributions were invested in capital could raise the saving rate moving the system toward the golden rule. His serendipity theorem states that at the goldenest golden rule rate of population growth, private saving would be exactly right to achieve the golden rule, with no public intervention needed.

Unfortunately, Samuelson had not checked the second order conditions for an optimum, and Deardorff pointed this out in a 1976 article to which Samuelson issued a gracious *mea culpe* and noted: "Professor Deardorff has shown that it can well be the case that the only root to my extremum condition corresponds to a minimum rather than a maximum of well-being."(1976a:516) His "Agreement and Evaluations" response to Deardorff is a valuable contribution in itself, dealing with the role of the ignored natural resource constraints, declining fertility, and other important issues.

A quite different comment by Arthur and McNicoll (1978) is extremely interesting. This brief and compact comment has been fundamentally important in pointing the way to combining rich and realistic demography and descriptive empirics of economic behavior over the life cycle with neoclassical growth theory. Rather than seeking the optimal population growth rate or birth rate, in this comment they set out to "evaluate the implications of a change in population growth assuming that production and consumption are spread realistically over a continuous-age lifecycle and that people are treated as individuals from birth, with a welfare criterion that reflects their expected lifetime utility of consumption." They are concerned with assessing the real world consequences of high versus low fertility. It turns out that this was also what Samuelson had in mind. He (1976a:519) tells us that "the original purpose of the 1975 exercises was to shed possible light on the vital public policy questions involved in social security, Modigliani lifecycle saving-investing patterns, apparent declining patterns of demographic fertility, and normative discussions of family planning." Later he refers specifically to the consequences of the decline in US fertility following the Baby Boom.

Arthur and McNicoll compare the lifecycle utility achieved for different population growth rates across golden rule steady states, assuming that a social planner makes the kinds of adjustments described above by Samuelson to keep the economy on a golden rule path. The setup presented earlier in this chapter was derived in part from this Arthur and McNicoll comment, and using the earlier notation, I will summarize a result that is closely related to their main result:

(0.9)
$$\frac{dC(n)/dn}{C(n)} = A_c - A_{yl} - \frac{k}{c}.$$

where C is lifetime survival-weighted consumption of an individual at birth, *k* is capital per capita (rather than the capital/labor ratio). An increase in the population growth rate requires that more output be saved to provide capital for the more rapidly growing labor force, as reflected in the last term on the right. But an increased population growth rate also alters the relative numbers of people at different ages, changing the relative numbers of supporters and dependents as in the consumption-loan model. These changes require many adjustments throughout the economy, including in the amount consumed at each age. Whatever the age pattern of changes in consumption, the left hand side of the equation tells us the proportional change in the present value of lifetime consumption that results from a change in n, and the right hand side tells us that there is an intergenerational transfer effect given by the difference in the average ages, and a capital dilution effect, and these are additive. As Arthur and McNicoll tell us, once we include the consumption of children in the story, it is no longer necessary that the effect of more rapid growth on dependency, as expressed in the difference in average ages, will be beneficial. It may well be that more rapid population growth, leading to higher proportions of children in the population and lower proportions of elderly, is itself costly.

From Figure 1 we know that the difference in average ages is negative in low and middle income countries, so that the intergenerational transfer effect reinforces rather than offsets the capital dilution effect. In rich countries, however, it may be either positive or negative with a difference in the range of plus or minus five years, at most.

Lee and Mason et al (2014) examined the effect of fertility on the steady state levels of consumption under golden rule given the empirically observed age profiles in 40 countries (based on (1.9), and concluded that in most cases lower fertility would be beneficial. The exceptions were countries with fertility below 1.5 births per woman (Total Fertility Rate). In general, the effects of fertility variation in rich industrial nations were small since actual fertility is typically fairly close to the maximizing level. As Samuelson put it (1976a:519) "Paradoxically, Deardorff's minimizing g^t is as encouraging as Samuelson's maximizing g* to reduce the fears that declining population growth makes old-age security more difficult. In the neighborhood of an extremum, whether it is a minimum or a maximum, a function does not change much."

The take-away message here is that although low fertility does raise the costs per worker of old age dependency, and in rich countries raises total dependency even when the dependency costs of children are taken into account, these rising dependency costs are largely offset by reduced need for saving to provide capital for a growing labor force, so the net effect on lifetime consumption is quite small.

Wealth, Social Security, and Capital

In 1975 when Samuelson published his (flawed) paper on the optimal rate of population growth, he also published a paper on the optimum Social Security program. In his analysis, the reason for Social Security is not to compensate for people's inability to save for their retirement. He assumes people save rationally for retirement. In his analysis Social Security is a policy instrument whereby the government can adjust the national saving rate and capital stock to reach the golden rule steady state growth path. He already showed, with his Serendipity Theorem, that if the population growth rate happens to be at its optimal level, say n*, then private life cycle saving will also be exactly right to achieve the golden rule capital intensity, and no government intervention is required. However, if population growth n^1 is more rapid than n* then he finds that the saving rate will be too low to achieve the corresponding golden rule path for n^1. In this case, he suggests that a more than fully funded mandatory Social Security program can move the economy to the golden rule path, raising lifecycle consumption. Alternatively, if fertility is very low and the population growth rate n^2 drops below its optimum, then the saving rate and capital intensity may be too high, and the rate of interest may drop below n^2. This economy is on an inefficient growth path with too much capital. Now a Pay-As-You-Go Social Security program can substitute for some of private saving, reducing the national saving rate and moving the economy to the golden rule path with less capital and higher interest rates.

We hear similar ideas today in discussions of "secular stagnation" with interest rates dropping to zero or below, while investors still do not borrow out of fear that slow population growth will reduce future demand for their products. Secular stagnation is said to be caused in part by declining rates of technological progress, but also by slowing population growth rates and population aging. One of the proposed policy interventions is to raise Pay-As-You-Go public transfers to the elderly, thereby reducing the motivation for private saving, reducing capital intensity, and raising interest rates, much as Samuelson suggested.

These ideas can be addressed using the accounting machinery introduced earlier. Recall that the aggregate demand for wealth, W, is the amount of wealth that the whole population would have to hold in order for each generation to achieve its future consumption plans given its future expected labor earnings. Since debt and credit must exactly cancel in a closed economy, net credit must be zero and cannot be a store of aggregate wealth. As we saw earlier, socially constructed transfer patterns, public or private, can be a store either of aggregate wealth or its negative, aggregate debt, with transfer wealth denoted T. The other possible store of aggregate wealth is capital K: houses, machines, roads, and so on. So aggregate wealth, W, and be held either as capital or as transfer wealth (Willis, 1988; Lee, 1994):

(0.10) $$W = K + T$$

Here government policy can manipulate T, for example by expanding or reducing the generosity of a Pay-As-You-Go Social Security program. This then leads to a change in K, which will alter interest rates, wages, and W, leading to a series of changes before arriving at a different equilibrium that could be closer to the golden rule.

We also have (1.5), (1.7) and (1.9). Combining these, we arrive at the striking result across golden rule steady states:

(0.11) $$d \ln C(n) / dn = T/c$$

This tells us that the proportional effect of a slight increase in the population growth rate on lifetime consumption equals aggregate transfer wealth relative to per capita consumption. If transfer wealth is positive, which is the case when the net direction of transfers is from younger to older ages as in modern welfare states with the strong net transfers to the elderly for pensions and health care, then an increase in the population growth rate can raise lifetime consumption. When the net direction of transfers is downwards, as in most lower and middle income countries, more rapid population growth is costly. But note also that when transfer wealth is zero, a small variation in the population growth rate has no effect on lifetime consumption, indicating that the growth rate is there at the optimal level, and that the golden rule has been achieved without the sorts of government intervention that Samuelson discusses. This result is closely related to the Serendipity Theorem.

Conclusion

In his work on population, as elsewhere, Samuelson stripped problems to their essential elements and went straight to the heart of the matter both through his models and through his interpretations of the results. At the same time, he was wary of analyses that oversimplified so that results depended on omission of a key constraint, as with natural resources in relation to both population growth and economic growth. Although the work reviewed here is very abstract and mathematical, according to Samuelson himself it was motivated by a desire to address issues of compelling policy importance. This was true of his work on fertility cycles in relation to the Baby Boom and Bust, and also in relation to the fertility decline and population aging in industrial nations and their consequences for economic growth, dependency, and lifetime consumption. No other mainline economist in the 20^{th} century contributed so broadly and deeply to population theory for humans, for other species, for evolution, and for the significance of age structured life cycles in economic equilibrium and growth.

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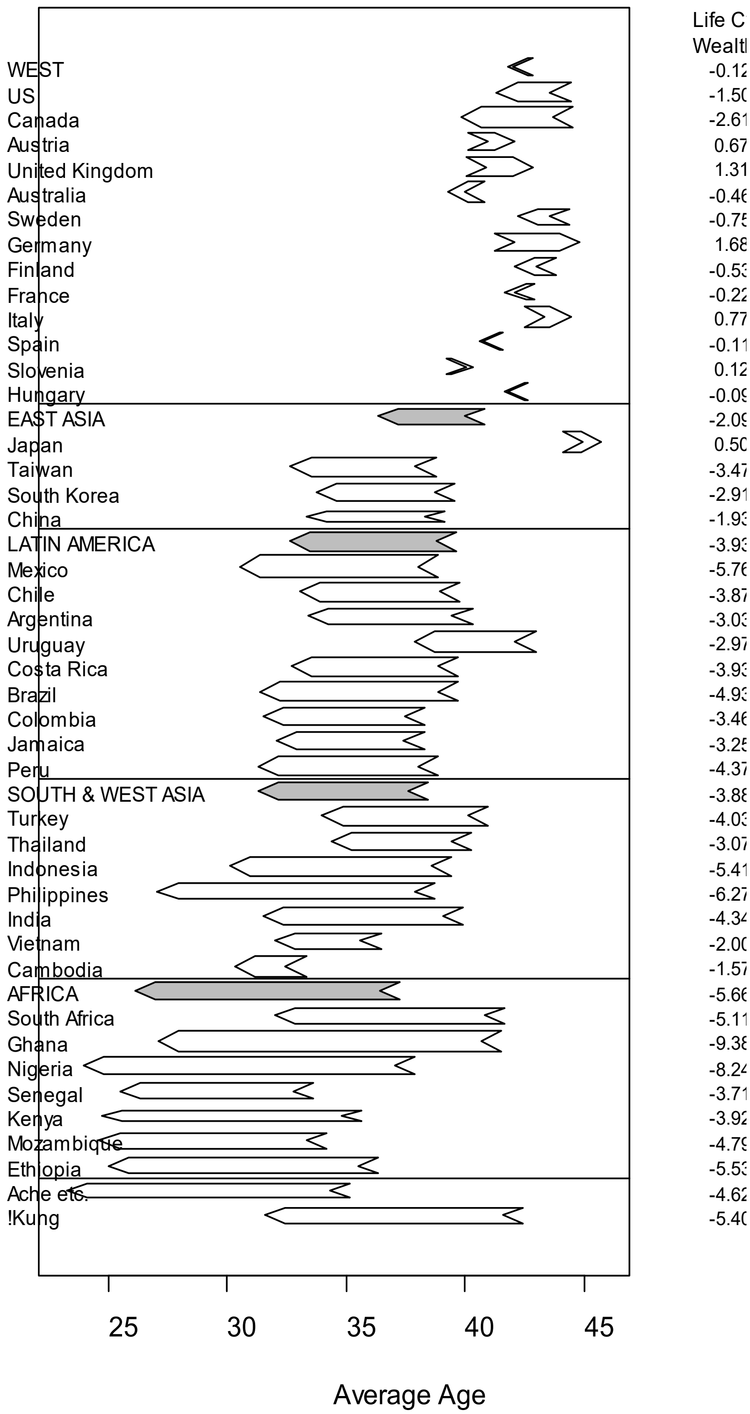
Figure 1. Arrows showing the average age of earning labor income (at the tail) and the average age of consuming (at the head) weighted by actual population age distributions, for 39 countries around the world in a recent year, and for two contemporary hunter gatherer/forager groups.



Source: For countries of the world, the data come from the National Transfer Accounts project (NTA) at NTAccounts.org. The hunter-gatherer data come from Kaplan (1994) for the Ache and Howell (2010) for the !Kung.