IZA Institute of Labor Economics

Initiated by Deutsche Post Foundation

DISCUSSION PAPER SERIES

IZA DP No. 15127

Demographic Changes, Labor Supplies, Labor Complementarities, Calendar Annual Wages of Age Groups, and Cohort Life Wage Incomes

Bjarne S. Jensen
Peder J. Pedersen
Ross Guest

MARCH 2022

I Z A Institute
of Labor Economics

Initiated by Deutsche Post Foundation

DISCUSSION PAPER SERIES

IZA DP No. 15127

Demographic Changes, Labor Supplies, Labor Complementarities, Calendar Annual Wages of Age Groups, and Cohort Life Wage Incomes

Bjarne S. Jensen
University of Southern Denmark

Peder J. Pedersen
Aarhus University and IZA

Ross Guest
Griffith University

MARCH 2022

ISSN: 2365-9793

IZA – Institute of Labor Economics

Schaumburg-Lippe-Straße 5–9
53113 Bonn, Germany

Phone: +49-228-3894-0
Email: publications@iza.org

www.iza.org

ABSTRACT

Demographic Changes, Labor Supplies, Labor Complementarities, Calendar Annual Wages of Age Groups, and Cohort Life Wage Incomes

This paper analyzes the impact on age group wage differentials in a setting of imperfect labor substitution at different ages (years) of working life. We examine the wage prospect of assuming medium, high, and low levels of fertility during the population projection period (2020-2090). Main focus is on comparisons of selected Calendar year Age wage profiles and the comparisons of selected Cohort Lifetime wage profiles. The analytical results come from applying a CRESH Labor Aggregator to Age-group Labor supplies with a parametric calibration to register based micro data for Denmark. The results show Calendar year wage effects and Cohort wage effects from ageing that will not exist without non-zero Labor Complementarity elasticities, and are new contributions demonstrating the economic effects of large/small generations and cohort sizes. The impact of cohort size on the lifetime wage profile of its own cohort does depend on sizes of other cohorts, which are affected by the fertility rates underlying many cohorts. Hence, economic advantages of being a small cohort depend on fertilities and the sizes of many other existing cohorts.

JEL Classification: J1, O4, E2

Keywords: labor substitution, CRESH, demographic cohorts, lifetime wage incomes

Corresponding author:
Peder J. Pedersen
Department of Economics and Business Economics
Aarhus University
Fuglesangs Alle 4
DK-8210 Aarhus V
Denmark
E-mail: ppedersen@econ.au.dk

1 Introduction

Demographic changes (projections) affect the *Population Age Distribution* as well as size and *Age Composition* (absolutely and relatively) of the available *Labor supplies* from the relevant working age groups. This paper address economic implications of *imperfect substitution* and *complementarity* of the *Labor services* from different Labor *Age groups*.

The standard assumption in demographic *macro* modelling is that the aggregate labor variable is a simple sum of the *homogeneous* labor services of different age groups - which implies perfect substitution and *same wage*. This means that the influence (size effect) of aggregate labor supply by an *increase* in workers of a *particular* age-group is *not* affected by the *Age distribution* (relative number) of workers already in the labor force. For example, when younger workers are becoming relatively scarce, it makes sense to allow for their *age-specific* contribution to an Aggregate measure (**Aggregator**) of Labor supplies. Hence we allow for *labor heterogeneity* by specifying a parametric CRESH[1] labor aggregator. This analytic **Labor Aggregator** function has implications for *relative wages* of both younger and older workers. In particular, if labor income is higher at younger ages and lower at older ages, then total *Lifetime* wage income of some generations (or cohorts) may be higher, while others are lowered. Various wage impacts are defined and calculated with lower/higher fertility of current and future calendar year generations.

In this paper, the *purpose* is to offer an *analytic globally regular* labor supply function (CRESH **Labor Aggregator** formed by any finite number of *labor supply* variables) with an *empirical applications* (parameter calibrations) to Danish *micro data* - and potential use for any country, where application of the principle of imperfect labor substitution is warranted. Our *focus* is next on investigating various *micro* and *macro implications* of projected **demographic** changes in this **century** (2010-2100) upon *relative* and *absolute annual wages* of *11 five-year Age groups* of *all* working ages (15-69) in *selected Calendar years* and then give the *lifetime labor incomes* of some *proper defined Cohorts*.

Among the *main new results* with our analytically extended CRESH wage model formulations are the extensive CRESH demonstrations (scenarios) of *comparative wages* (relative/absolute) of *all age groups* for some calendar **years** (t) in period (2020-2090),

[1]CRESH stands for Constant Ratio of Elasticities of Substitution, Homothetic, (Hanoch, 1971).

as well as obtaining the *Life time* wage incomes for *selected* **Cohorts** (Generations) of different *sizes*, entering the Labor market in the **year**, T= 2010, 2015, 2020, 2030, 2035.

Design and estimations of Labor aggregator (supply) functions have a long history. Only a short literature review is given here. Dougherty (1972, p.1110-16) discuss Labor aggregation structures based on 8 non farm *occupations* or 8 *educational* (length) categories. The aggregation functional forms are single-level CES functions or many *two-level* CES aggregations.[2] *Leontief* forms (fixed manpower requirements, $\sigma = 0$), *Linear* aggregation, $\sigma = \infty$), were extreme (invalid) forms, and *CD* function ($\sigma = 1$) implied too little scope for substitution (inappropriate for aggregating labor). CES function were seen as improvements on these special forms of aggregation. Chiswick (1985, p.503) adhered to CES with moderately *high elasticity* ($\sigma = 2.5$) between each pair of factors (including labor (*human capital*) of at least two quality levels of salaried employees). For US, UK, Canada, Card and Lemieux (2001, p.709,725) *estimated* (σ) in the *range*: 4-6 (1/0.23, 1/0.17) for two CES *subaggregates* (High School, College) of workers from 7 *age groups*. Recently, Guest and Parr (2020, p.509) used, $\sigma = 1$, $\sigma = 2$, for Australian CES labor aggregates of 11 age groups. However, long ago Berndt and Christensen (1973, p.407) proved that a consistent CES aggregators at all points in factor space is equivalent to *equality* and *constancy* of *all* Allen-Uzawa *partial* elasticities of *substitution* (AUES, σ_{ij}). Evidently, the substitution elasticities between many labor services of different *age-groups* have *never* been the same or strictly constant (independent of labor supplies) anywhere.

Clearly, more sophisticated aggregator functions than CES are then to be considered. But functional complexity must be restrained to preserve sensible theoretical and empirical robust patterns of the substitution elasticities (σ_{ij}). It is here that the CRESH function of Hanoch (1971, p.697) enter as a proper aggregator of different (*heterogeneous*) labor services, since it allows *relative substitution patterns* of σ_{ij} between services to be *preserved* (remain constant). Moreover, with our focus on the consequences for the *wage structure* of sizeable changes in the *age composition* of the (exogenous, demographic) *labor supplies*, we need to see for CRESH functions also the (dual) **partial complementarity** *elasticities* (c_{ij}), Sato and Koizumi (1973, p.47), which **link** the *relative* and *absolute*

[2]Bowles (1970, p.77) gave *Labor Supply Aggregates* with two-level CES functions of Sato (1967, p.202).

wage changes to variety in **Labor supplies of Age-groups**. These structure-analytic issues are probed jointly with CRESH calculations of (σ_{ij}) and (c_{ij}) in **Appendices A-B**.

Already Freeman (1979, p.301-303,313) estimated *complementarity elasticities* (c_{ij}) by the *Trans-Log* production function, using *CPS* (Current Population Survey) data tapes of individual (*Micro*) age-earnings (age-wage profiles). Our *Micro* (Personal Register) data on *Danish* labor *supplies* (annual full time *equivalents*) of *age groups* and annual *wages* are provided by *Statistics Denmark* (Department of Labor and Income).

The paper is organized as follows.

Section 2 presents in **Table 1** and **Figure 1** already known demographic trends in the *Age composition* of *Populations* in this century as the background for our economic analysis of relative wages and life time labor incomes. It describes *Labor supplies* from *Microlevel* (Register) data for year 2010 in **Table 2**, within a *Macro* framework - National Income Product Accounts (*NIPA*) for year 2010, shown in **Table 3**.

Section 3 presents the CRESH labor aggregator and the implications for age-wage profiles. It explains the methodology of calibrating the **CRESH** parameters to Labor *Micro* data of *2010* ; the calibrated CRESH model is validated on *Micro* data for *2013* in **Table 4**.

Section 4 calculates *demographic* - for Medium, Low, and High fertility from *Table 1* - *projected* Labor Supplies of eleven Age groups, spanning working life of 55 years (15-69). It applies the **CRESH** model for the projected Labor supplies by showing the *comparative wages* (relative/absolute) of *all age groups* for *calendar* **years** (t) in period (2020-2090). The *main results* are collected in **Tables (5a-5c)**, **Table 6**, and exhibited in **Fig. 2-6**.

Section 5 demonstrates the **CRESH** calculations of *Life time* wage incomes of *selected* **Cohorts**, *entering* the Labor market in *particular years* (T) of this century. The *main results* are explained and demonstrated by **Table 7** and illustrated in **Figures 7-12**.

Further micro and macro aspects of Labor aggregation and CRESH Age-wage profiles are discussed in section 6 in reference to the literature on Wage structure from 'Division of Labor' by labor of various levels of experience, skills - *Canonical Model*, Appendix C.

Section 7 offer final comments/suggestions for teaming up Demography and Economics.

Appendices (A,B,C) derive the basic CRESH Labor substitution elasticities and the *new* CRESH **complementarity** (Hicks-Sato) *elasticities* for *all* the Labor *age-group wages.*

2 Population age groups, Labor supplies and Wages

2.1 Demographic outlook, assumptions and future age groups

Let us briefly give the demographic outlook. Denmark, like most other developed countries, faces demographically further population ageing for some decades. **Table 1** shows Danish Population *age shares*, $\mathbf{n_i} = \mathbf{N_i}/\mathbf{N}$, of *three age* groups: children (0-14), working age (15-69), and old age (70+) - under **three** *Fertility* 'variants' (Medium, Low, High), cf. **Table 1a**, as published by the United Nations Population Division (United Nations, 2015). For the three age shares (n_i), males and females are combined. *Life expectancy* is the same under all fertility variants. The *'Medium'* variant projections assume that the Total Fertility *rate* **(TFR)** slightly and monotonically *increases* from **1.730** to **1.876**.

In the **Medium** variant scenario, the *Working* Population *share* ($\mathbf{n_{15-69}}$) declines monotonically to a *minimum* (0.521) in year 2050, after which it monotonically recovers to (0.715), similar to its present size. The 'Medium' variant share numbers (n_i) indicate an population *"ageing"* or "burden" problem for the next 20-40 years.

The **Low** variant numbers (n_i) suggest in contrast that population ageing or "*burden*" problems will occur *after 2050.* The **High** variant numbers (n_i) show a *remarkable stable* population composition *after 2025* - with even the old age (70+) share in balance.

When Fertility rate (TFR) is permanently less than **2.0**, there would be long-run tendency for the total size of population (N) to decline. However, if *Life Expectancy* is steadily *increasing*, then population (N) may still increase, despite (TFR) $<$ 2.0. Population $\mathbf{N(t)}$ for 2010-2100 in Medium Variant, Table 1, does *not decline* in *any* year.

The Danish **Population** N(2010)= 5.551 million. For *Medium* Variant, projected numbers are: N(2020)= 5.776, N(2030)= 6.003, N(2050)= 6.299, N(2100)= 6.838 million. *Low* Variant, Table 1, population eventually does *decline.* For Low Variant, projected numbers are: N(2020)= 5.732, N(2030)= 5.792, N(2050)= 5.603, N(2100)= 4.599 million. *High* Variant, Table 1, population certainly does *increase.* For High Variant, projected numbers are: N(2020)= 5.819, N(2030)= 6.214, N(2050)= 7.154, N(2100)= 9.843 million.

The *three* population *age* shares, $n_i = N_i/N$, define a *dependency rate* of *young*/children (0-14) to working age population : ($\mathbf{d_y}$), and an *old*/age (70+) dependency rate to working age population: $(\mathbf{d_o})$, and hence give the **total dependency ratio**: $(\mathbf{d})$, defined as:

$$d_y = \frac{n_{0-14}}{n_{15-69}} = \frac{N_{0-14}}{N_{15-69}}\ , \quad d_o = \frac{n_{70+}}{n_{15-69}} = \frac{N_{70+}}{N_{15-69}}\ ; \quad d = d_y + d_o\ ; \quad \frac{1}{1+d} \ = \ n_{15-69} \tag{1}$$

which, as calculated in **Table 1b**, are exhibited in **Fig. 1**. Note that in Table 1b, , e.g., *Medium* variant, **2010** : $1/(1+d) \ = \ (1/1.411) = \mathbf{0.709} \ = \ n_{15-69}$, (age share), **Table 1**. Thus, total dependency *ratio* $(\mathbf{d})$ is *uniquely* related to $\mathbf{n_{15-69}}$, i.e., the columns (d), **Table 1b**, tell essentially, for every variant, a *similar* story as $\mathbf{n_{15-69}}$ in **Table 1**, e.g. *Medium* variant, **2050** : *small*, $\mathbf{0.643} = n_{15-69}$, and *high* value of $\mathbf{d} = 0.555$; but monotonicity of $(\mathbf{d})$ in the *Low* variant seems more "*dramatic*" than the monotonicity of $\mathbf{n_{15-69}}$.

The projected *dependency ratios*, $\mathbf{d_y}$, $\mathbf{d_o}$, (1), are *dominated* by the *paths* of n_{15-69}, although n_{70+} exerts significant influence on (d_o) in the Low variant. It is projections of the dependency ratio, (d_o), that has attracted attention in the literature, Rojas (2005, p.466), Hu et al. (2000, p.117), Kitao (2015, p.38). When dependency ratio (d_o) is seen *redefined* as: $\bar{\mathbf{d}}_\mathbf{o} = N_{65+}/N_{15-64}$ (retirement age, 65), projected sizes of these numbers $(\bar{d}_o)$ appear in the literature more spectacular than $\mathbf{d_o}$ in **Table 1b** for *Denmark*: 2010-2100.

The Labour Force *Participation rate*, **LFP**, is defined by, $L_{15-69}/N_{15-69} \ = \ l_{15-69}$:

$$LFP \ = \ \frac{L_{15-69}}{N_{15-69}} \ = \ l_{15-69} \tag{2}$$

The **Support ratio** (L/N), defined as the ratio of total **Labor** force (Labor supply) (L) (**Employment**) to total **Population** $(N \ = \ N_{0-70+})$ is obtained from the *dependency ratio* (d), cf. (1), and the Labour Force Participation (**LFP**) rate, l_{15-69}, (2), as follows :

$$\frac{L}{N} \ = \ \frac{L_{15-69}}{N} \ = \ \frac{L_{15-69}}{N_{15-69}} \cdot \frac{N_{15-69}}{N} \ = \ l_{15-69} \cdot n_{15-69} \ = \ l_{15-69} \cdot \frac{1}{1+d} \tag{3}$$

Hence with e.g., **LFP** for 2010 : $l_{15-69} = 0.536$, we get by (3) for the *Medium variant*, the Support ratio in 2010 : $L/N = 0.536 \cdot 0.709 = \mathbf{0.38}$. Evidently, for a given, l_{15-69}, (constant LFP), the **Support** (Employment/Population) **ratio** (L/N), (3) gives - for every fertility variant - the *same scenario* as the projected Working Population share: $\mathbf{n_{15-69}}$ in **Table 1** - or *inversely* with the projected dependency ratio : $(\mathbf{d})$ in **Table 1b**.

The **rising** *dependency* ratio, (d_o), implies that the Danish *support ratio*, (L/N), **falls** from 0.38 (2010) to a level around 0.34 *after* 2050. The support ratios (L/N) are now declining in many countries and are expected to continuously *fall* in the years *until* 2050.

Table 1a. Fertility and Life Expectancy Assumptions : Denmark, 2010-2100				
	Medium variant Fertility*	**Low variant Fertility***	**High variant Fertility***	**Life Expectancy****
2010-2015	1.730	1.730	1.730	8.507
2015-2020	1.761	1.511	2.011	8.779
2020-2025	1.785	1.385	2.185	9.053
2025-2030	1.804	1.304	2.304	9.344
2030-2035	1.817	1.317	2.317	9.653
2035-2040	1.829	1.329	2.329	9.987
2040-2045	1.841	1.341	2.341	10.304
2045-2050	1.848	1.348	2.348	10.609
2050-2055	1.854	1.354	2.354	10.903
2055-2060	1.858	1.358	2.358	11.205
2060-2065	1.862	1.362	2.362	11.505
2065-2070	1.865	1.365	2.365	11.798
2070-2075	1.868	1.368	2.368	12.091
2075-2080	1.870	1.370	2.370	12.396
2080-2085	1.872	1.372	2.372	12.715
2085-2090	1.874	1.374	2.374	13.028
2090-2095	1.875	1.375	2.375	13.353
2095-2100	1.876	1.376	2.376	13.691

Notes:
Variants differ only with respect to fertility assumptions.
* Fertility refers to number of children per woman.
** Life expectancy at age 80 for both sexes combined (number of years).

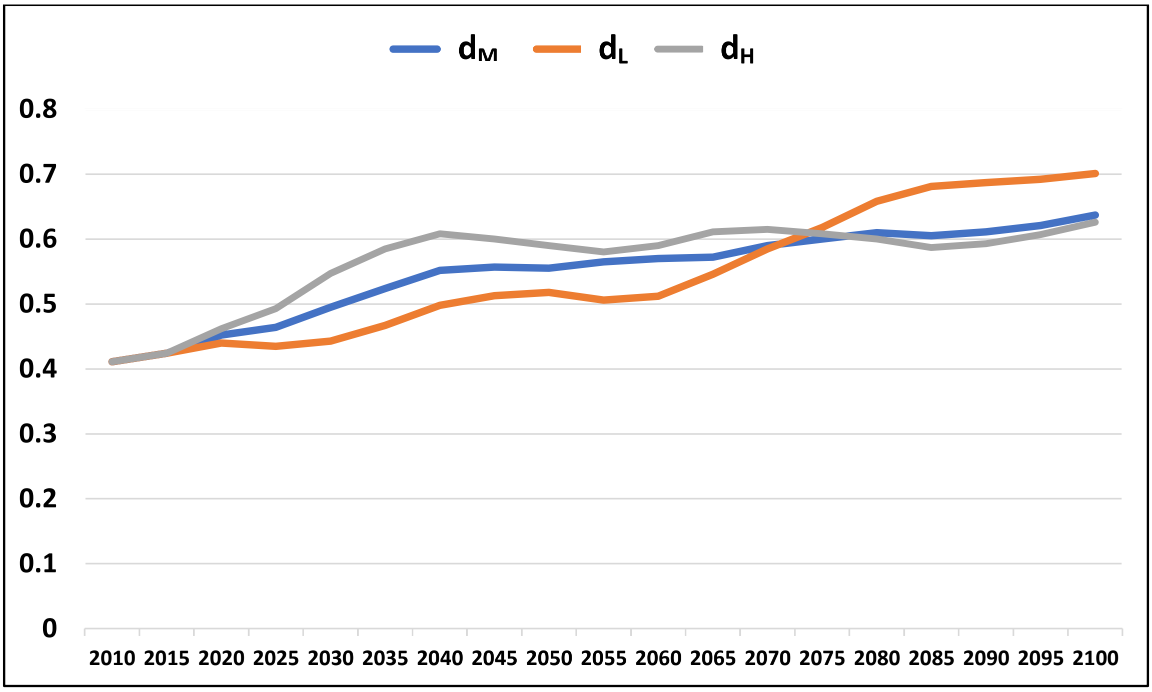
Table 1. Population Age Shares, $n_i = N_i/N$, i = 0-14, 15-69, 70+ : Denmark, 2010-2100									
	Medium Variant (n_i)			**Low Variant (n_i)**			**High Variant (n_i)**		
	0-14	**15-69**	**70+**	**0-14**	**15-69**	**70+**	**0-14**	**15-69**	**70+**
2010	0.180	0.709	0.111	0.180	0.709	0.111	0.180	0.709	0.111
2015	0.173	0.702	0.125	0.173	0.702	0.125	0.173	0.702	0.125
2020	0.163	0.689	0.148	0.157	0.695	0.149	0.169	0.684	0.147
2025	0.160	0.683	0.157	0.143	0.697	0.160	0.176	0.670	0.154
2030	0.165	0.669	0.166	0.135	0.693	0.172	0.193	0.646	0.160
2035	0.168	0.656	0.176	0.132	0.682	0.186	0.201	0.631	0.168
2040	0.167	0.644	0.189	0.130	0.667	0.202	0.200	0.622	0.178
2045	0.164	0.642	0.194	0.128	0.661	0.211	0.195	0.626	0.180
2050	0.161	0.643	0.196	0.125	0.659	0.216	0.193	0.629	0.178
2055	0.160	0.639	0.201	0.120	0.664	0.216	0.197	0.633	0.170
2060	0.161	0.637	0.202	0.116	0.661	0.223	0.204	0.629	0.167
2065	0.163	0.636	0.201	0.114	0.646	0.239	0.209	0.621	0.170
2070	0.162	0.629	0.209	0.113	0.631	0.256	0.208	0.619	0.173
2075	0.160	0.625	0.215	0.112	0.618	0.270	0.204	0.622	0.174
2080	0.157	0.621	0.222	0.110	0.603	0.287	0.201	0.625	0.174
2085	0.155	0.623	0.222	0.108	0.595	0.297	0.200	0.630	0.170
2090	0.155	0.621	0.224	0.106	0.593	0.301	0.202	0.628	0.170
2095	0.155	0.617	0.228	0.105	0.591	0.304	0.203	0.622	0.175
2100	0.155	0.611	0.234	0.105	0.588	0.307	0.203	0.615	0.182

Source: United Nations (2015) 'World Population Prospects, 2015 Revision', United Nations, New York, 2015.
Total population (both sexes combined) by five-year age group.

The graphics of the *Danish dependency ratio* (d) in **Table 1b** is exhibited in **Figure 1** for the *three* UN demographic[3] *variants* (Medium, High, Low).

[3]The **declining** *fertility* in *recent decades* and hence the *falling* dependency *ratio*, (d_y), have in several countries *dominated* the *rising* (d_o), such that **Support ratios** (L/N), (3), in some countries have in certain **periods until 2010** actually **increased** - and been called "*demographic (fertility) dividends.*" We shall in **Table 6** see a few economic illustrations of this "dividend" in the Danish *Medium* fertility *variant* for *some years* after the '*minima*' of the 'Working Age Population' *share*, $\mathbf{n_{15-69}}$, in **2050**. Some illustrations of US *dependency ratios* and *Support ratios* are seen in, Cutler et al. (1990, p.5,8).

Fig. 1. *Danish Dependency ratio -* **d** *- for Medium, Low, High fertility, 2010-2100.*



<u>Source</u>: Total dependency ratio (**d**), (1), with the numbers from **Table 1b**.

Table 1b. Population Dependency Ratios, (1) : d_y , d_o , d : Denmark, 2010-2100

	Medium Variant			Low Variant			High Variant		
	d_y	d_o	d	d_y	d_o	d	d_y	d_o	d
2010	0.254	0.157	0.411	0.254	0.157	0.411	0.254	0.157	0.411
2015	0.246	0.178	0.424	0.246	0.178	0.424	0.246	0.178	0.424
2020	0.237	0.215	0.452	0.226	0.214	0.440	0.247	0.215	0.462
2025	0.234	0.230	0.464	0.205	0.230	0.435	0.263	0.230	0.493
2030	0.247	0.248	0.495	0.195	0.248	0.443	0.299	0.248	0.547
2035	0.256	0.268	0.524	0.194	0.273	0.467	0.319	0.266	0.585
2040	0.259	0.293	0.552	0.195	0.303	0.498	0.322	0.286	0.608
2045	0.255	0.302	0.557	0.194	0.319	0.513	0.312	0.288	0.600
2050	0.250	0.305	0.555	0.190	0.328	0.518	0.307	0.283	0.590
2055	0.250	0.315	0.565	0.181	0.325	0.506	0.311	0.269	0.580
2060	0.253	0.317	0.570	0.175	0.337	0.512	0.324	0.266	0.590
2065	0.256	0.316	0.572	0.176	0.370	0.546	0.337	0.274	0.611
2070	0.258	0.332	0.590	0.179	0.406	0.585	0.336	0.279	0.615
2075	0.256	0.344	0.600	0.181	0.437	0.618	0.328	0.280	0.608
2080	0.253	0.357	0.610	0.182	0.476	0.658	0.322	0.278	0.600
2085	0.249	0.356	0.605	0.182	0.499	0.681	0.317	0.270	0.587
2090	0.250	0.361	0.611	0.179	0.508	0.687	0.322	0.271	0.593
2095	0.251	0.370	0.621	0.178	0.514	0.692	0.326	0.281	0.607
2100	0.254	0.383	0.637	0.179	0.522	0.701	0.330	0.296	0.626

<u>Source</u>: The dependency ratios, $\mathbf{d_y}$, $\mathbf{d_o}$, **d**, are defined in (1) and calculated by **Table 1**.

The overall dependency ratio, **d**, (1), is *rising* for each fertility scenario in **Figure 1**. But in the *high* fertility scenario, the dependency ratio (**d**) is 'stationary' in the 50 years from *2040 to 2090*. In the *low* fertility scenario, we find a significant *increase* in the dependency ratio (**d**) from *2060* onwards, as the delayed impact of prior low fertilities.

As well-known, a *Fertility* rate of **2.1** (children per woman) on average is usually necessary for reproduction of population levels; increasing Life Expectancy modifies the requirement. As mentioned above Total Danish Population size, $\mathbf{N(t)}$, never declines, but slowly *increases* during projection *period* 2020 − 2090 under *Medium* Variant of **Tables (1a, 1)**, cf. **Table 6** (Row 4) below - that also shows that $\mathbf{N(t)}$ *declines* after 2040 in the *Low* Variant, and clearly $\mathbf{N(t)}$ *increases* (nearly doubles by 2090) for the *High* Variant.

Having presented the United Nations projections of the **evolution** of the Danish *demographic structure* 2010-2100 in **Tables (1,1b,6)**, **Fig. 1**, with the *general concepts* and *terminology*, (1-3) - similar descriptions apply to any UN country - we restate (for later use) overall **LFP**, (2), in terms of **11** *age-specific* labor participation rates, $l_i = \frac{L_i}{N_i}$:

$$LFP = \frac{L_{15-69}}{N_{15-69}} = l_{15-69} = \sum_{i=1}^{11} \frac{L_i}{N_i} \cdot \frac{N_i}{N_{15-69}} \equiv \sum_{i=1}^{11} l_i \cdot \widetilde{n}_i \; ; \; L_i(t) = l_i \cdot N_i(t), \; i = 1,,11 \quad (4)$$

With proper chosen $\mathbf{l_i}$ as *exogenous* parameters, we can derive *age-specific* Labor supplies in any *calendar* year (t), $\mathbf{L_i(t)}$, (4), from the evolutions of, $N_i(t) = n_i(t) \cdot N(t)$, and $N(t)$.

Introducing imperfect [4] labor *substitution/complementarity* between *age-specific* Labor group supplies $\mathbf{L_i(t)}$, (4), significantly *changes* the *relative wages* within the Total Labor force (supply), $\mathbf{L_{15-69}(t)}$, over time. We will use a suitable **Labor** economic **model** *analytically designed* to generate/explain such **Age** Group **Wage Differentials**. Section 3 presents a **CRESH** model with distinct *parameters* for labor age group supplies.

[4] The assumption of perfect substitution of labor among age groups has been challenged, tested empirically and relaxed in a variety of modelling approaches, Prskawetz et al. (2008), Guest (2007), Creedy and Guest (2007), Guest and Shacklock (2005), Hamermesh (1993), Lam (1989).

2.2 Labor supplies of age groups, Micro wage data, and NIPA

To apply the CRESH labor *aggregator* for empirically analyzing age-wage profiles, the CRESH parameters must be properly related to specific *labor supplies* and *wage data*. Here we use *Micro* and *Macro* data for Denmark in the year 2010 - to be explained and shown in **Tables 2-3**. Similar Micro data 2013 are used for CRESH aggregator validation.

As provided from the United Nations World Population Prospects, 2015 Revision, total **Population** (both sexes) by *five year age groups* (United Nations, 2015), the Danish Population numbers, N_i (column 2, **Table 2**) are : The *demographic* sizes of our *eleven* 5-year *working* age groups (15-69), and the *young* (0-14) and *old* age (70+) groups, i.e., the *absolute* sizes of the age groups in *2010*, corresponding to *age shares* (n_i) in **Table 1**.

Within the *eleven* 5-year working age groups, the **oldest** N_i, (65-69), soon fully retired, are **born** in (1941-1945). In (1945,1946), the number of *births peaked* with (95-96.000). The *post-war* (1946-1950), *generation* are seen in $\mathbf{N_i}$ (60-64). Birth rates started to slowly decline in the 1950's ; the Danish economy was stagnating until 1957, and net *emigration occurred*, as can be seen from the N_i (55-59) numbers, which also partly reflect a *negative* **'echo'** of the smaller depression year generations of 1930's. In contrast, a *positive* **'echo'** of two *war-postwar* generations above and prosperous *full-employment* years of 1960's are reflected in sizes $\mathbf{N_i}$ (45-49), $\mathbf{N_i}$ (40-44) of the *two generations*, (1961-1965), (1966-1970). The European oil-shock *recession* and unemployment *years*, (1976-1980), (1981-1985), are reflected in the *small Danish* numbers of the, N_i (30-34), N_i (25-29).

From the beginning of 1990's, revenues from North Sea oil - as in UK and Norway - contributed to remove deficits of Danish international and public sector accounts. *Child benefits* were subsequently increased; significant *immigration* also began to matter in these years. They are explanatory population elements of a *turn-around*, seen in sizes of both N_i (20-24), and the **youngest** age group, N_i (15-19), **born** in (1991-1995).

We must next explain the **Labor** *supplies* used and their associated *wages* in 2010. The *Labor age group* numbers (Labor years) $\mathbb{L}_i$ (column 3, **Table 2**) are Danish *full time* workers (*equivalents*, **1924 hours**) - with *age distribution* ($\lambda_{\mathbf{i}}$), (col. 4), and their *average annual wages* ($\mathbf{w_i}$), (column 6). These Microlevel data (personal register) were provided to the authors by Statistics Denmark. These *Register data*, however, were excluding agriculture, fishery, and all firms with less than 10 full-time employees.

We calibrate our model to *National Accounting* data for Denmark in **2010**, which implies that the *sum* of $\mathbf{L_i}$ (col. 5) must equal *aggregate employment* (**Table 3**, row 1) of **2112472** full time workers (*Labor years*). We use the *age-specific* labour *fractions* ($\lambda_\mathbf{i}$) of *Register* data (column 4) to gross up the values of $\mathbb{L}_i$ such that the *total* of $\mathbf{L_i}$ (col. 5) is equal to : $\mathbf{L = 2112472}$. The *age-specific* wages $\mathbf{w_i}$ in **Table 2** (col. 6), of Register data are multiplied by the adjusted (Total) Labor numbers, $\mathbf{L_i}$, (col.5), and summed to give the *aggregate Wage Bill*, which is **924.3** Billion DKK (col. 8). The *aggregate Annual wage*, **w**, per unit of **L** is then found by dividing the aggregate Wage Bill (col. 8) by **L**, which gives : $\mathbf{w\ (2010) = 437552}$ DKK $\equiv$ $\mathbf{W_A}$; cf. (41), **Tables (5a, 5b)**.

Table 2. Age-specific Data: Labor groups, Wages, Wage shares, Participation rates - Denmark 2010

1 Age (i)	2 N_i	3 $\mathbb{L}_i$	4 λ_i	5 L_i	6 w_i	7 w_i/w_4	8 $w_i L_i$	9 ε_i	10 $l_i = L_i/N_i$
15-19	353109	21138	0.0143	30199	187005	0.4554	5.647	0.0061	0.0855
20-24	331419	77504	0.0524	110731	273220	0.6653	30.254	0.0327	0.3341
25-29	310515	117177	0.0792	167411	358262	0.8724	59.977	0.0649	0.5391
30-34	347261	170826	0.1155	244059	410668	1.0000	100.227	0.1084	0.7028
35-39	388101	202853	0.1372	289816	449679	1.0950	130.324	0.1410	0.7468
40-44	408902	214618	0.1451	306624	471118	1.1472	144.456	0.1563	0.7499
45-49	405079	211983	0.1434	302860	472491	1.1505	143.099	0.1548	0.7477
50-54	366102	188472	0.1275	269270	471381	1.1478	126.929	0.1373	0.7355
55-59	350020	169527	0.1147	242203	461729	1.1243	111.832	0.1210	0.6920
60-64	368451	88874	0.0601	126974	478708	1.1657	60.783	0.0658	0.3446
65-69	309369	15626	0.0106	22325	483248	1.1767	10.789	0.0117	0.0722
Total 15-69	3938328	1478598	1.0000	2112472	437552		924.317	1.0000	0.5364
0-14	997084			0	0				0
70+	615547			0	0				0
Total	5550959			2112472	166515		924.317		0.3806

Source: UNITED NATIONS (UN), see **Table 1** ; STATISTICS DENMARK (Department of Labor and Income), Copenhagen.

Column 1: **Age groups**, i = 1, …,11; i = 1: 15-19, …, i = 11: 65-69 ; i = 12: 0-14, i = 13: 70+.
Column 2: Population (totals) in age groups; UN Population Data.
Column 3: *Full time* workers (Annual *equivalents*, **1924 hours**), Labor services in **Labor years ;** *Microlevel* (personal register) *data*.
Column 4: Labor age group distribution - Fractions, λ_i , **same** in column 3 and column 5.
Column 5: **Total** full time workers in labor **age** groups, $L_i = \lambda_i L$, (**L = 2112472** = Total full time workers) ; stat.bank, **DB07, ERHV1.**
Column 6: Average annual **wages** of labor **age** groups (w_i) in column 3 and 5; **w = 437552** DKK = **924.317** Billion DKK / **2112472**.
Annual wage income per capita, wL/N = 924.317 Billion / 5550959 = 166515 DKK (Danish Kingdom Kroner).
Column 7: Relative annual wages, **age** group **wage profile -** generated by the *Microlevel (*personal register) *data* in **column 6.**
Column 8: Total wage incomes of **age** group, (***i***)**,** Billion DKK; (stat.bank, **DB07**, ERHV1, **Total** wage sum : **930.286** Billion DKK).
Column 9: Age group wage income shares ε_i (shares in the total wage bill, 924.317 Billion DKK).
Column 10: Labor participation rates (**LPR**) of age groups - derived from column 2 and column 5.

	Table 3. National Income Accounts - Data for Denmark, 2010			
	Descriptions	**Symbols**	**Values**	
1	Total (equivalent) full time workers	*L*	2112472	Labor years
2	Average (aggregate) wage per labor year (man-year)	*w*	437552	DKK
3	Total wage incomes	*wL*	924.3	Billion DKK
4	Net capital (rental) incomes	*rK*	303.8	Billion DKK
5	Net Factor Incomes (NFI) – Net Domestic Value Added	*Y = wL + rK*	1228.1	Billion DKK
6	Capital consumption/depreciation	δK	318.1	Billion DKK
7	Gross Factor Incomes (GFI) – Gross Domestic Value Added	*GFI*	1546.2	Billion DKK
8	Net capital stock	*K*	5741.5	Billion DKK
9	Net capital-output ratio	*v = K/Y*	4.67	
10	Net capital-labour ratio	*k = K/L*	2.72	Million DKK
11	Average labour productivity	*y = Y/L*	581972	DKK
12	Depreciation rate	$\delta = \delta K/K$	0.055	percent
13	Net real interest rate	*r = rK/K*	0.053	percent
14	Gross real interest rate	$r + \delta$	0.108	percent
15	Wage share of net factor income, *w/(Y/L)*	$\varepsilon_L = wL/Y$	0.752	
16	Capital share of net factor income, *r/(Y/K)*	$\varepsilon_K = rK/Y$	0.248	
17	Factor compensation, Asset income (net), to rest of world	*O*	29.6	Billion DKK
18	Net National Income , in factor prices	*Y+O = NNI*	1257.7	Billion DKK
19	Gross National Income , in factor prices	*C+Tr+S = GNI*	1575.8	Billion DKK
20	Consumption (private + public) , in factor prices	*C*	1104.9	Billion DKK
21	Transfers to rest of world, net	*Tr*	36.6	Billion DKK
22	Gross National Saving	*S*	434.3	Billion DKK
23	Gross Domestic Investment, in factor prices	*I*	331.3	Billion DKK
24	Balance of payment, current account, Asset accumulation	*S - I = BP*	103.0	Billion DKK
25	Consumption ratio, NNI	*C/NNI*	0.879	
26	Consumption ratio, NFI	*C/Y*	0.900	
27	Consumption per capita, in factor prices	*C/N*	199350	DKK
28	Net National Income per capita, in factor prices	*NNI/N*	226573	DKK
29	Annual wage income per capita, in factor prices	*wL/N*	166515	DKK
30	Support Ratio	*L/N*	0.3806	
31	Net Factor Income (NFI) per capita, Y/N = (Y/L)(L/N)	*Y/N*	221499	DKK
32	Annual wage income-consumption ratio	*wL/C*	0.837	
33	Gross national saving rate	*S/GNI*	0.276	
34	Net national saving rate	$(S - \delta K)/NNI$	0.092	
35	Gross domestic investment rate	*I/GFI*	0.214	

Source : **Rows** 1-3, *Microlevel* (register) data from ***Table 2***. **Rows** 5-9, 12, 17-26, *Macro* (aggregate) data from **NIA** (National Income Accounting) : *Statistical Ten-Year Review,* (**STR**) *2015,* p. 101-102, 104-105, 120 ; STATISTICS DENMARK, Copenhagen.
Row 1: Full time **Labor years**, *L*= 2.112.472 = 4064.4 Million labor hours (1 **Labor year** = 37 hours per week x 52 = **1924 labor hours**) ; STR (2015, p.121) gives in **NIA** : 3606.6 Million labor hours for 2010.
Row 3: STR (2015, p.104) gives in **NIA** : Wage Sum = **953.7** Billion DKK – a bit more (½ %) more than, ***wL* = 924.3** Bill. DKK.
Row 4: ***rK*** = Row 5 (***Y***) – Row 3 (***wL***) ; STR (2015, p.104) gives in **NIA** : Net Capital Income = 274.4 – a bit less than, ***rK*= 303.8.**.
Rows 5-7: **STR** (2015, p.104), with same ***Y* = 1228.1,** as in Row 5. Row 8: STR (2015, p.128) . Row 17: STR (2015, p.102).
Row 18: Row 5 + Row 17.
Row 19: STR (2015, p.104) gives : *GNI* in market prices = *GNI* in factor prices + indirect taxes = 1578.8 + 252.5 = 1828.3 Billion DKK.
Row 20: STR (2015, p.104) gives : *C* in market prices = *C* in factor prices + indirect taxes = 1104.9 + 252.5 = 1357.4 Billion DKK.
STR (2015, p.102) gives : Indirect commodity (production) taxes/subsidies, net : 248.2 + 4.3 = 252.5 Billion DKK.
Row 21: STR (2015, p.102) ; Row 22: STR (2015, p.104); Row 23: STR (2015, p.103); Row 24: STR (2015, p.106).
Rows 25-26: Derived from rows above; Row 27: ***C/N*** = *(C/Y) (L/N)* (*Y/L*) = (0.9) (0.3806) 581972 = **199350** DKK, cf. **Table 2**.

Labour Force Participation (**LFP**) rate (4), *Support ratio* (3) - cf. **Table 2**, col.10,1 - are:

$$LFP = \frac{L_{15-69}}{N_{15-69}} = l_{15-69} = 0.5364 \ ; \ \frac{L}{N} = \sum_{i=1}^{11} l_i \cdot n_i = \sum_{i=1}^{11} \frac{L_i}{N_i}\frac{N_i}{N} = l_{15-69} \cdot n_{15-69} \quad (5)$$

$$\frac{L}{N} = l_{15-69} \cdot n_{15-69} = 0.5364 \cdot 0.7095 = 0.3806 \ ; \ \frac{Y}{N} = \frac{Y}{L} \cdot \frac{L}{N} \ ; \ \frac{C}{N} = \frac{C}{Y} \cdot \frac{Y}{N} \quad (6)$$

Support ratio L/N (5-6) is n_i-weighted age-specific, l_i. Support ratio: a multiple of **LFP**. The *per capita sizes* of National Income, Consumption ratios, Y/N, C/N, and their *decomposition* in (6) are seen in NIPA, **Table 3** (row 31,27) - summarized in **Table 3a**.

Thus **Micro** based *employment* - full time equivalents - and *wage* data in **Table 2** (col. 5,6,8), correspond exactly to **Macro** (National Income) data in **Table 3** (Row 1-3, 29-31), for calendar *year*, 2010.[5] Short version of **Table 3** is seen in **Table 3a** - template to **Table 6** - as calendar year summary of labor model results in **Tables (5a, 5b)**.

Table 3a. Population Age Groups, Labor Supply, Support Ratios, Incomes per capita: Denmark			
2010	1.	$N_{15\text{-}69}$	3938328
	2.	$N_{0\text{-}14}$	997084
	3.	N_{70+}	615547
	4.	*N = Total*	5550959
	5.	$L = L_{15\text{-}69}$	2112472
	6.	$L/N_{15\text{-}69}$	0.5364
	7.	*L/N*	0.3806
	8.	*WL Bill.*	924.317
	9.	*WL/N,*	166615
	10.	*Y/N*	221499

Table 3a/Table 6, Rows 1-4 show - in *absolute* quantitative form (Total numbers) for calendar years - the consequences of the *demographic changes* described in **Tables (1,1a,1b)**. Rows 5-7 show, respectively, the absolute sizes of the *total* labor force, $L(t)$, (supply), the sizes of **LFP** (t) rate, (5), and the sizes of Support Ratio, **L(t)/N**(t), (6). Row 8 shows the Total **Wage** *Income* of $L(t)$, [All age groups, in calendar year (t)], working in any year (t) with **11** *age-specific* participation rates : l_i (t) = l_i (2010), **Table 2** (col. 10). Row 9 shows the Total **Wage** *Income* per capita, $N(t)$, which - with the *macroeconomic* structure, technology levels, productivity conditions, (Y/L), (K/L), of **Table 3**, (2010) - is *equivalent* to : $\mathbf{L(t)}/\mathbf{N}(t) \cdot \mathbf{w(2010)}$ = Support ratio $\cdot \mathbf{W_A}$[6]. Row 10 shows **National Income** per capita, $Y(t)/N(t)$, which is here a simple *proportionality* of Row 9, cf. (6).

The labor market equilibrium model with CRESH *Aggregator* functions - underlying all results in **Tables 4-7** - must now be established and justified, theoretically and empirically, in Section 3 and Appendix A : Labor *Substitution* and *Complementarity.*

[5]The year 2010 - as benchmark for our projected population variants, annual labor supply and wage income comparisons - is chosen for various reasons. It takes several years before the final revision of National Income Accounting (NIA) is completed. The Financial Crisis years (2008-2009) were unsuitable as benchmark years. For year 2010 the final revision came out in 2015. The processing of the Micro register databases for corresponding employment and wage data for 2010 began after NIA revision 2015. We cannot wait for getting reliable revised NIA for 2015 or later - as our model benchmark year.

Remark. The *overall* sizes of LFP = l_{15-69} in (5), **Tables** (2,4,6) look small ; the *age-specific* LFP, l_i, **Table 2** show that l_1, l_{11} give small l_{15-69} (Age group 65-69, l_{11}, is *seldom* included in reported LFP).

[6]As will be explained by Factor (Labor) cost (income) functions and *duality* theory in *Appendix* A.

3 CRESH Labor supply, Relative wages, Annual wage

Hanoch (1971, p. 697) introduced a *globally regular* CRESH *implicit* production (aggregator) function, $F(Y, X_1, X_2, \ldots, X_M) = 0$. Our CRESH function, $\mathbf{F}(\mathbf{L_A}, \mathbf{L_1}, \mathbf{L_2}, \ldots, \mathbf{L_M})$, as seen in equation (7), is *homogeneous* of degree **zero** - and **F** determines **implicitly** the **Labor Aggregate** variable, $\mathbf{L_A}$, (*Total* Labor *Supply*), from the *distinct* (heterogeneous) Labor services, $(L_1, L_2, ..., L_M)$, (M Labor *Supplies*), as stated in the expression:

$$F\left(L_A, L_1, L_2, ..., L_M\right) \;=\; \gamma \sum_{i=1}^{M} \alpha_i \left[\frac{L_i}{L_A}\right]^{\rho_i} - 1 \;=\; 0 \tag{7}$$

with

$$\gamma > 0;\;\; \forall i:\; \alpha_i > 0,\;\; \sum_{i=1}^{M} \alpha_i = 1;\;\; \forall i:\; 0 <\; \rho_i \leq 1 \;\; \mathit{or}\;\; \rho_i < 0 \tag{8}$$

Labor services (flows), $(\mathbf{L_1}, \mathbf{L_2}, ..., \mathbf{L_M})$, may be *measured* in hours, working-weeks, or labor years. As in **Table 2-3**, we use as Labor **unit**: *Labor years* ; the total flow variable $(\mathbf{L_A})$ is also measured in Labor years. Thus *ratios*, $(\frac{L_i}{L_A})$, in (7) are *unit-free* (pure numbers), implying, too, that all **parameters** in (8) are *unit-free* (pure numbers).

For $\forall i$: $\rho_i = \rho$, we get CES functions by (7-8), and CD as the limit function $(\rho = 0)$,

$$\rho_i \;=\; \rho \;:\;\; L_A = \gamma^{\frac{1}{\rho}} \left[\sum_{i=1}^{M} \alpha_i L_i^{\rho}\right]^{\frac{1}{\rho}} \;;\;\; \rho = 1,\; L_A = \gamma \sum_{i=1}^{M} \alpha_i L_i \;;\;\; \rho = 0,\; L_A = \bar{\gamma} \prod_{i=1}^{M} L_i^{\alpha_i} \tag{9}$$

Parameter *restrictions* (8) ensure that **CRESH** equation (7) represents a **unique** implicit **Labor Aggregator** function, $\mathbf{L_A} = \mathbf{f}\left(\mathbf{L_1}, \mathbf{L_2}, ..., \mathbf{L_M}\right)$, that is *homogeneous* of degree **one**, and is globally regular, i.e. for all $\mathbf{L_i} > 0$, $\mathbf{f}\left(,\ldots,\right)$ is *positive*, *non-decreasing*, **concave**, with a negative semi-definite *Hessian* matrix, $\frac{\partial^2 \mathbf{f}}{\partial \mathbf{L_i}\, \partial \mathbf{L_j}}$.

$$\forall L_i \;> 0 \;:\;\; L_A = f\left(L_1, L_2, ..., L_M\right) \;> 0 \;;\;\; \frac{\partial f}{\partial L_i} > 0 \;,\;\; \frac{\partial^2 f}{\partial L_i^2} < 0 \;;\;\; L_A \;=\; \sum_{i=1}^{M} \frac{\partial f}{\partial L_i}\, L_i \tag{10}$$

The CRESH function, $F\left(L_A, L_1, L_2, ..., L_M\right)$, in (7) has the first-order derivatives,

$$\frac{\partial F}{\partial L_i} = \frac{\gamma \alpha_i \rho_i \left(L_i/L_A\right)^{\rho_i - 1}}{L_A} \;,\;\;\; i = 1, ..., M \;;\;\;\; \frac{\partial F}{\partial L_A} \;=\; -\,\frac{\gamma \sum_{i=1}^{M} \alpha_i \rho_i \left(L_i/L_A\right)^{\rho_i}}{L_A} \tag{11}$$

Marginal *contributions* of L_i to L_A : $\frac{\partial L_A}{\partial L_i} = \frac{\partial f}{\partial L_i}$, and *marginal* rates of *substitution* (**MRS**) are given by implicit differentiation of $F\left(L_A, L_1, L_2, ..., L_M\right)$, i.e. we get by (10-11):

$$\frac{\partial L_A}{\partial L_i} = \frac{\partial f}{\partial L_i} = -\,\frac{\partial F/\partial L_i}{\partial F/\partial L_A} = \frac{\alpha_i \rho_i \left(L_i/L_A\right)^{\rho_i - 1}}{\sum_{i=1}^{M} \alpha_i \rho_i \left(L_i/L_A\right)^{\rho_i}} > 0 \,,\;\; i = 1, ..., M \tag{12}$$

$$\frac{d\,L_j}{d\,L_i} = MRS = \frac{\partial f/\partial L_i}{\partial f/\partial L_j} = \frac{\alpha_i \rho_i}{\alpha_j \rho_j} \frac{(L_i/L_A)^{\rho_i - 1}}{(L_j/L_A)^{\rho_j - 1}}, \quad i \neq j \tag{13}$$

The elasticities, $E\,(L_A, L_i)$, *shares* (ε_i), add up to 1 by the degree of homogeneity in (10),

$$\varepsilon_i \;=\; E\,(L_A, L_i) \;\equiv\; \frac{\partial L_A}{\partial L_i}\frac{L_i}{L_A} \;=\; \frac{\alpha_i \rho_i\,(L_i/L_A)^{\rho_i}}{\sum_{i=1}^{M} \alpha_i \rho_i\,(L_i/L_A)^{\rho_i}} \quad ; \quad \sum_{i=1}^{M} \varepsilon_i = 1 \tag{14}$$

Relative wages - relative factor prices, (13) - must reflect their **MRS**. Hence CRESH *relative wages*, $(\frac{w_i}{w_j})$, CRESH *relative* wage income *shares*, $(\frac{\varepsilon_i}{\varepsilon_j} = \frac{w_i L_i}{w_j L_j})$, become by (13-14):

$$\frac{w_i}{w_j} \;=\; \frac{\alpha_i \rho_i}{\alpha_j \rho_j} \frac{(L_i/L_A)^{\rho_i - 1}}{(L_j/L_A)^{\rho_j - 1}} \;=\; \frac{\alpha_i \rho_i}{\alpha_j \rho_j} \frac{L_i^{\rho_i - 1}}{L_j^{\rho_j - 1}}\, L_A^{\rho_j - \rho_i}, \; i \neq j \;; \quad \frac{\varepsilon_i}{\varepsilon_j} \;=\; \frac{\alpha_i \rho_i}{\alpha_j \rho_j} \frac{(L_i/L_A)^{\rho_i}}{(L_j/L_A)^{\rho_j}} \tag{15}$$

These CRESH expressions emphasize the *relative wage effects* of particular labor *supplies* (pair), L_i, L_j, the *substitution* parameters, ρ_i, ρ_j, and the relative *intensity* parameters, α_i, α_j. Via **Total** variable L_A, *all* variables L_i, and *all* parameters in (7) affect $(\frac{w_i}{w_j})$, (15).

The special cases of (15) for the CD-CES family (9) become $(1 - \rho \,=\, \frac{1}{\sigma})$:

$$CD: \; \frac{w_i}{w_j} = \frac{\alpha_i}{\alpha_j} \frac{L_j}{L_i} \;; \;\; CES: \; \frac{w_i}{w_j} = \frac{\alpha_i}{\alpha_j} \left[\frac{L_i}{L_j}\right]^{\rho - 1} \;; \;\; Linear: \; \frac{w_i}{w_j} = \frac{\alpha_i}{\alpha_j} \;; \; i \neq j \tag{16}$$

If (9) takes the linear form, *relative wages* (16) depend *only* on relative intensity *parameters*, α_i, α_j, whereas, L_i, L_j, *also* affect CD, CES, (16). On CRESH aggregator, see Conlon (1993); for discussions of empirical estimation of CRESH, see Weiss (1977, p.765).

Changes in relative wages, w_i/w_j, (15), are *smaller*, the *higher* is the value of ρ_i. Intuitively, the *more flexible* a labor supply is (higher value of ρ_i), the *smaller* change in its *relative wage* is required to *clearing* the *labor markets* (supply-demand equilibrium) for the given change in supply of the labor service, L_i.

The CRESH elasticity of the wage ratio, (w_i/w_j), with respect to the labor supply (L_i), is simply obtained from, (15), (14), (by elasticity rules for composite functions):

$$E\,[\,\frac{w_i}{w_j}, L_i\,] \;=\; \rho_i - 1 + (\rho_j - \rho_i)\,\varepsilon_i < 0 \;\; ; \;\; E\,[\,\frac{w_j}{w_i}, L_i\,] \;=\; 1 - \rho_i + (\rho_i - \rho_j)\,\varepsilon_i > 0 \tag{17}$$

where ε_i is *labor income share* of L_i. Thus by (17), *increasing* L_i will always *decrease* the **CRESH** *relative wage* of L_i; but the higher ρ_i is, the smaller is the percentage decline in (w_i/w_j); a higher ρ_j has a similar effect on diminishing the decline in (w_i/w_j) as ρ_i. Moreover, a larger L_i will always *increase* the CRESH *relative wages* of L_j (other labor groups compared to L_i); the higher ρ_i is, the larger is the relative increase in (w_j/w_i); the effect of higher ρ_j gives a smaller increase in (w_j/w_i), as a result of larger L_i.

By (15) - and using the same elasticity rules above - we also here note that,

$$E\,[\,\frac{\varepsilon_i}{\varepsilon_j}, L_i\,] \;=\; \rho_i + (\rho_j - \rho_i)\,\varepsilon_i > 0 \;\;;\;\; E\,[\,\frac{\varepsilon_j}{\varepsilon_i}, L_i\,] \;=\; -\rho_i + (\rho_i - \rho_j)\,\varepsilon_i < 0 \qquad (18)$$

Thus, in *contrast* to their *relative wages* in (17), the *relative labor shares* of L_i in (18), always *increases* with larger L_i ; moreover, the **CRESH** *relative* labor *income shares* of the *other* labor groups L_j *decline*, when L_i is increased.

The labor *services*, $(L_1, L_2, ..., L_M)$, can refer to **any** disaggregation of *labor supply.* Our services relate to *labor* **age groups** ; hence CRESH (15) relative wages will represent : *Age Group* **Wage Differentials** - to be linked up to *demographic* labor *supply projections.*

Since logically, $E\,(\frac{w_i}{w_j}, L_i\,) = E\,(w_i, L_i) - E\,(w_j, L_i)$, $E\,(\frac{\varepsilon_i}{\varepsilon_j}, L_i\,) = E\,(\varepsilon_i, L_i) - E\,(\varepsilon_j, L_i)$, we should *embed* the pairwise CRESH *relative annual wage* relations and **ratio** elasticities (13-18) into a **complete** CRESH framework of **comparative statics** for the **absolute** '*own-price*', $E\,(w_i, L_i)$, '*cross-price*', $E\,(w_j, L_i)$, *wage* elasticities, factor *share* (distributional) elasticities, and hereto labor *substitution* and labor *complementarity* elasticities.

All these elasticities and the **basic economic** implications of CRESH function (7-8) are revealed and derived below by using **duality** theory for *implicit* CRESH *Aggregator* function, (10) : $f\,(L_1, L_2, ..., L_M)$, *Wage Cost* function, $C\,(w_1, w_2, ..., w_M, L_A)$, and *Wage Income* function, $W\,(L_1, L_2, ..., L_M, W_A)$. Our **new** and important expressions for *labor* **complementarity** *elasticities* (c_{ij}) are derived for CRESH, (7), (10), in **Appendix B**[7].

3.1 CRESH model calibration and validation : 2010, 2013

In **Table 4** (Col. 2,5,6c) is collected the **2010 data** of wage shares, (ε_i), relative wages, (w_i/w_4), w_i, i=1,..,M. The **intensity** (weight) parameters $(\alpha_\mathbf{i})$ in CRESH Labor supply (7) - are obtained by calibrations, as described below ; cf. Guest and Jensen (2016, p.30).

From (15), we get :

$$\frac{\alpha_i}{\alpha_j} \;=\; \frac{\varepsilon_i}{\varepsilon_j}\frac{\rho_j}{\rho_i}\frac{(L_j/L_A)^{\rho_j}}{(L_i/L_A)^{\rho_i}} \;=\; \frac{\varepsilon_i}{\varepsilon_j}\frac{\rho_j}{\rho_i}\frac{(L_j)^{\rho_j}}{(L_i)^{\rho_i}}\,L_A^{\rho_i-\rho_j} \quad ;\; i \neq j \qquad (19)$$

In (19), 2010 wage **shares** (ε_i), i=1,..,M, are known (Col.2), and so by making particular **assumptions** (choices) of the *substitution* parameters (ρ_i), i=1,..,M in Col.3, and by using also the 2010 **data**, $\mathbf{L_i}$, i=1,..,M, $L_A = L$, from **Table 2** (Col.5), the **ratios** of the intensity parameters, $(\alpha_\mathbf{i}/\alpha_\mathbf{j})$, can then be derived (**calculated**) from the equation (19).

[7] *Ratio* elasticity, $E\,[\,\frac{w_i}{w_j}, L_i\,]$, in (17), comes from (87-88) & *complementarity elasticities*, c_{ij}, (78-79).

Table 4. Wage income shares, (*i*), CRESH parameter values, (*i*), Relative wages, (*i*), Absolute wages, (*i*) : Data and Model - Denmark 2010, 2013

2010								2013									
1 Age (*i*)	**2** ε_i data	**3** ρ_i	**4** α_i	**5** w_i/w_4 data	**6a** w_i/w_4 model	**6b** w_i model	**6c** w_i data	**7** N_i	**8** $\mathbb{L}_i$	**9** λ_i	**10** L_i	**11** w_i data	**12** w_i/w_4 data	**13** w_i/w_4 model	**14** w_i model*	**15** w_i model**	**16** $l_i = L_i/N_i$ data
15-19	0.0061	0.8	0.0316	0.4554	0.4557	187061	187005	358224	17514	0.0123	25929	187815	0.4432	0.4572	193754	201755	0.0724
20-24	0.0327	0.8	0.0598	0.6653	0.6650	272987	273220	359625	67404	0.0472	99498	272254	0.6425	0.6612	280194	291765	0.2767
25-29	0.0649	0.7	0.0756	0.8724	0.8727	358235	358262	322222	111329	0.0780	164425	364345	0.8598	0.8538	361782	376723	0.5103
30-34	0.1084	0.7	0.0970	1.0000	1.0000	410492	410668	328063	150950	0.1057	222817	423743	1.0000	1.0000	423743	441243	0.6792
35-39	0.1410	0.6	0.1070	1.0950	1.0953	449623	449679	371971	188695	0.1322	278680	466584	1.1011	1.0824	458667	477609	0.7492
40-44	0.1563	0.5	0.1134	1.1472	1.1471	470892	471118	388543	199024	0.1394	293857	493182	1.1639	1.1397	482955	502901	0.7563
45-49	0.1548	0.5	0.1131	1.1505	1.1512	472556	472491	427351	217491	0.1523	321051	503375	1.1879	1.0875	460827	479858	0.7513
50-54	0.1373	0.6	0.1090	1.1478	1.1491	471699	471381	374370	189628	0.1328	279944	495068	1.1683	1.1007	466395	485656	0.7478
55-59	0.1210	0.7	0.1089	1.1243	1.1253	461908	461729	353381	172363	0.1207	254437	484135	1.1425	1.0789	457161	476041	0.7200
60-64	0.0658	0.8	0.1078	1.1657	1.1664	478818	478708	341481	92507	0.0648	136599	499596	1.1790	1.1188	474077	493656	0.4000
65-69	0.0117	0.8	0.0768	1.1767	1.1765	482944	483248	352035	20949	0.0146	30777	509897	1.2033	1.0738	455026	473818	0.0874
Total 15-69	1.0000		1.0000			437552	437552	3977266	1427854	1.0000	2108014	459463			441241	459463	0.5300
0-14								977596									
70+								647766									
Total								5602628			2108014	172875					0.3763

Source: Year 2010, see **Table 2** ; Year 2013 : UNITED NATIONS ; STATISTICS DENMARK (Department of Labor and Income), Copenhagen.

Column 1: Age groups, $i = 1, \ldots, 11$; $i = 1$: 15-19, …, $i = 11$: 65-69.
Column 2: Age group wage income shares (wage cost shares in the total wage bill), cf. column 9, Table 2.
Column 3: Substitution parameters in CRESH function equation.
Column 4: Intensity parameters in CRESH function equations, $L_A = L = 2112472$.
Column 5: Actual relative wages (Age group annual wage profile), given by data, Table 2.
Column 6a: CRESH Model (optimal) Relative annual wages (Age group annual wage profile), 2010 : $\gamma = 4.378$, $L_A = L = 2112472$.
Column 6b: CRESH Model Absolute annual wages (Age groups) with, W (2010) = 437552, and with calculated w_4 (2010) = 410492 = (w_4 tilde).
Column 6c: The observed Annual wages 2010 : Data, cf. column 6, Table 2.
Column 7: Population (totals) in age groups, *Statistical Ten-Year Review* (STR), STR (2013, p.19).
Column 8: Full time (annual equivalent) workers, Labor services, (Labor years), *Micro* (individual, register) *data*.
Column 9: Labor age groups, relative sizes - fractions (λ_i) - obtained from column 8.
Column 10: Total full time workers, Labor services, $L_i = \lambda_i L$; 2013 : ($L = 2108014$; stat.bank, DB07, ERHV1, full-time employees, total).
Column 11: Average annual wages of labor age groups; Total Wages, $WL = 968.555$ Billion DKK ; $W = 459463$; (stat.bank, DB07, ERHV1, Annual wage sum, $WL = 964.998$ Billion DKK).
Column 12: Actual relative annual wages ('Age annual wage profile'), given by data from column 11.
Column 13: CRESH Model (optimal) Relative annual wages ('Age annual wage profile'), by w_i/w_4 ; 2013 : $\gamma = 4.365$, $L_A = L = 2108014$.
Column 14: CRESH Model Absolute annual wages of labor age groups, $w_i = w_4$ x column 13, with observed w_4 (2013) = 423743 - giving : $WL = 930.142$ Billion ; $W = 441241$.
Column 15 CRESH Model Absolute annual wages of labor age groups, $w_i = w_4$ x column 13, with calculated w_4 (2013) = 441243 - giving : $WL = 968.555$ Billion ; $W = 459463 =$ W (2013) = Data.
Column 16: Labor participation rates (LPR) of age groups - derived from columns (7,10).
Remark: As mentioned in Note 5, both the *Overall* LFP and some of the *Age-specific* LFP, $l_i = L_i/N_i$ in **Table (2,4)** (column 10,16) are smaller than often reported as LFP rates.
Labor Force in LFP (ILO statistics) consists of : Employees, and Unemployed. L_i in **Table (2,4)**, (column 5,10), are *Full-time equivalents*. Unemployment rate, 2010-13: 6 %, (approx. 100.000 persons).

By next *summing* the equation (19), and using, $\sum_{i=1}^{M} \alpha_i = 1$, cf. (8), we have,

$$\sum_{i=1}^{M} \frac{\alpha_i}{\alpha_j} \equiv \frac{1}{\alpha_j} \equiv \sum_{i=1}^{M} \frac{\varepsilon_i}{\varepsilon_j} \frac{\rho_j}{\rho_i} \frac{(L_j/L_A)^{\rho_j}}{(L_i/L_A)^{\rho_i}} = \frac{\rho_j}{\varepsilon_j} \Big[\frac{L_j}{L_A}\Big]^{\rho_j} \sum_{i=1}^{M} \frac{\varepsilon_i}{\rho_i \, (L_i/L_A)^{\rho_i}} \equiv \frac{\rho_j}{\varepsilon_j} \Big[\frac{L_j}{L_A}\Big]^{\rho_j} \mu \quad (20)$$

Thus the **size** of α_j is *determined* by the **RHS** expression of (20). Next (19-20) give :

$$\alpha_i = \frac{\varepsilon_i}{\varepsilon_j} \frac{\rho_j}{\rho_i} \frac{(L_j/L_A)^{\rho_j}}{(L_i/L_A)^{\rho_i}} \alpha_j = \frac{\varepsilon_i}{\rho_i} \Big[\frac{L_A}{L_i}\Big]^{\rho_i} \frac{1}{\mu} \,, \quad i = 1, ..., M \quad (21)$$

Hence **all absolute** values of α_i *parameters* are obtained by (21) - [with μ as seen in (20)].

By this **calibration** procedure, (21), such associated values of 11 CRESH **intensity** parameters, $(\alpha_{\mathbf{i}})$, **Table 4**, (Col.4), for 2010 can - with L_i, i=1,..,M, $L_A = L$, from **Table 2** - be *calculated* for *any* assumptions (*pattern*) of these 11 *parameters*, ρ_i, Col.3.

The actual **selected** CRESH *substitution* parameters $\rho_{\mathbf{i}}$ in **Table 4**, (Col.3) were determined as follows. An *initial* set of 11 (ρ_i) determines 11 (α_i), as described by (19) and (21). The *aim* is to find a *pattern* of ρ_i which generates CRESH *relative* wages (15) by $(\frac{w_i}{w_4})$, (j=4), (22) that *best fit*, (Col.6a), the *actual relative* wages, (2010), $(\frac{w_i}{w_4})$, (Col.5).

$$\frac{w_i}{w_4} = \frac{\alpha_i \rho_i}{\alpha_4 \rho_4} \frac{L_i^{\rho_i - 1}}{L_4^{\rho_4 - 1}} L_A^{\rho_4 - \rho_i} \,, \quad i = 1, ..., M \qquad \Leftrightarrow \qquad \frac{w_i}{w_4} = \frac{\alpha_i \rho_i}{\alpha_4 \rho_4} \frac{{\lambda_i}^{\rho_i - 1}}{{\lambda_4}^{\rho_4 - 1}} \quad (22)$$

Various patterns of ρ_i have been tested in this way for *Australia* as discussed in Guest & Jensen (2016). The best fit for *Denmark* is found to be the approximate **U-shape** *pattern* of ρ_i, shown in **Table 4**, (Col.3). This U-shape pattern of ρ_i implies that *middle age workers*, who have relatively *low values* for ρ_i have a mix of labor attributes ('qualities') that make them *harder* to *substitute* (replace) [lower ρ_i give smaller **substitution elasticities**, σ_{ij}, (49)] than the younger or older workers. This pattern has also important consequences for **wages**, relative and absolute [lower ρ_i give larger labor **complementarity elasticities**, c_{ij}, (57), for *age group* (i), and so group (i) have larger *annual wage elasticities*, $E(w_i, L_j) = \varepsilon_j \, c_{ij}$, (86), and hence gain larger wage increases by bigger labor supplies of other age groups L_j]. The *combined* set (sizes) for ρ_i, α_i, Col.3-4 fitted best **2010** : (Col.5, 6a); or (Col.6b, 6c) by (28-29), λ_i (22), RHS , cf. footnotes 8-9 below.

To **validate** the calibrated year 2010 CRESH parameter *values*, $(\rho_i, \, \alpha_i)$, **Table 4**, Col.3-4, we corroborate these parameter sizes $(\rho_i, \, \alpha_i)$ upon another data set, year 2013. Thus the **2013 data** seen in the six columns, **Table 4**, Col.7-12, correspond (with same content/explanations) exactly to earlier six columns for year 2010, **Table 2**, Col.2-7.

In order to *validate* the calibrated parameters outside the base year (2010), we insert the *calibrated* (**2010**) *values*, ρ_i, α_i, (Col.3-4), into our formula of *relative wage*, (22), together with using the observed **2013 data** for : L_i, i=1,..,M, $L_A = L$, (Col.10). Thus columns (Col.3-4,10) give by (22) the CRESH *results* for relative wages, $\frac{w_i}{w_j}$, (j=4), **Col.13** for year 2013[8] - to be compared with *observed* relative wages, $\frac{w_i}{w_j}$, (j=4), **Col.12** for 2013. Apart from age groups (60-64, 65-69), the **Col.12-13** are concurring pretty well for all age groups. So the calibrated CRESH **parameters**, ρ_i, α_i, (Col.3-4), are essentially **confirmed** (validated) on the **new data** set (**2013**).

By (22), only the **relative wage** numbers ($\frac{w_i}{w_4}$) were calculated for 2010 og 2013. But how to get the **absolute** sizes of the *Annual wages* (w_i) for *Ages*, i=1,..,M, in **Table 4** ?
Absolute wages. *Total Wage Income* for *all* Age groups (i) is by definition, **wL**, i.e,

$$wL = \sum_{i=1}^{M} w_i L_i \; ; \quad w_4 = \frac{wL}{L_4 + \sum\limits_{i \neq 4}^{M} \frac{w_i}{w_4} L_i} \; ; \quad w_i \; = \; w_4\,(2013) \cdot \frac{\alpha_i \rho_i}{\alpha_4 \rho_4} \frac{(L_i)^{\rho_i - 1}}{(L_4)^{\rho_4 - 1}} \, L_A^{\rho_4 - \rho_i} \qquad (23)$$

Dividing **LHS** of (23) by w_4, and using $[\frac{w_i}{w_4}]$, $i \neq 4$, rearranging, gives w_4, stated above. With w_4 (23) allows all w_i for 2013 to be calculated by **RHS** (23), by using *observed*, $w_4\,(2013)$ = 423743, **Table 4** (Col.14). However, by $w_4\,(2013)$ as 'scaling factor' for w_i, (23), generates for L_A = L (2013)= 2108014, the *Total* Wages : wL (2013) =930142 Billion, *Average* Annual wage, w (2013)= 441241. But Col.11, 15, give for $L_A = L\,(2013)$ the **actual** *Total* Wage **sum** : **wL** (2013) = 968555 Billion, *Average Annual* wage, w (2013)= 459463 $\equiv$ $\mathbf{W_A}$ (2013), i.e., $\mathbf{W_A}$(2013) gives by $w_4 = \widetilde{w_4}$ as in (24) a **consistent** *scaling wage* of $\frac{w_i}{w_4}$ to use in computing **absolute annual** wages ($\mathbf{w_i}$) for age, i=1,..,M, cf. Col.15:

$$w_i \; = \; \widetilde{w_4}\,(2013) \cdot \frac{\alpha_i \rho_i}{\alpha_4 \rho_4} \frac{(L_i)^{\rho_i - 1}}{(L_4)^{\rho_4 - 1}} \, L_A^{\rho_4 - \rho_i} \; , \; i\text{=1,..,}M \quad ; \quad \widetilde{w_4}\,(2013) = \frac{W_A(2013)\, L}{L_4 + \sum\limits_{i \neq 4}^{M} \frac{w_i}{w_4} L_i} \qquad (24)$$

All w_i (24) were still calculated with *chosen* α_i and ρ_i parameters from **2010**, **Table 4**. By using (22), (24), we have a *consistent* CRESH formula for **absolute** *Age wage* ($\mathbf{w_i}$) *calculations* to be applied *any year* (t) - also with $\mathbf{W_A}$, $\mathbf{w_i}$, defined by (25), (26) below.

[8]CRESH γ, (7), a "total productivity" (efficiency) *parameter* was *not* involved in *relative wages*, (15). For given values of α_i, ρ_i, *Table 4* (Col.3-4), the γ *size* can be adapted so that *aggregate variable*, (7), L_A = L (Total Labor force, Labor supply) = L(t)= $\sum\limits_{i=1}^{M} L_i$; for t=2010, 2013, see γ, **Table 4** (Col.6,13). Such γ values are to be used for *any year*, if as in *all* **Tables 4-7**, L_A = L(t)= $\sum\limits_{i=1}^{M} L_i\,(t)$.

3.1.1. From dual CRESH Labor Cost function (52), or dual **Wage Income** function (81) in **App.B**, we have for CRESH - $\mathbf{F}(\mathbf{L_A}, \mathbf{L_1}, \mathbf{L_2}, \ldots, \mathbf{L_M}) = 0$, (7) ; $\mathbf{L_A} = \mathbf{f}\,(\mathbf{L_1}, \mathbf{L_2}, ..., \mathbf{L_M})$, **Aggregator** (10) - the *basic duality* relations, cf. (12), (14-15), (52), (67-68), (81), (84):

$$W_A\, L_A = \sum_{i=1}^{M} w_i L_i \equiv wL \equiv W \equiv C \equiv c\,(w_1, w_2, , , w_M) L_A\,;\ W_A = c\,(w_1, w_2, , , w_M) \quad (25)$$

$$w_i \;=\; w_i\,(L_1, L_2, ..., L_M, W_A) = \frac{\partial W\,(L_1, L_2, ..., L_M, W_A)}{\partial L_i} = W_A\,\frac{\partial L_A\,(L_1, L_2, ..., L_M)}{\partial L_i} \quad (26)$$

$$= W_A \cdot \frac{\partial f\,(L_1, L_2, ..., L_M)}{\partial L_i} = W_A\,\frac{\alpha_i \rho_i\,(L_i/L_A)^{\rho_i - 1}}{\sum_{i=1}^{M} \alpha_i \rho_i\,(L_i/L_A)^{\rho_i}} = W_A\,\frac{\alpha_i\,\rho_i\,\lambda_i^{\rho_i - 1}}{\sum_{i=1}^{M} \alpha_i\,\rho_i\,\lambda_i^{\rho_i}} \quad (27)$$

$\mathbf{W_A}$ (25): *Arithmetic Average* of all *money age wages* ($\mathbf{w_i}$) - or 'shadow values', (26-27).

For demographic *projection period*, t : **2015-2090**, we *don't* have - as (24) in **Table 4** - *empirical* values of $\mathbf{W_A}\,(t)$. Throughout the projection period the **exogenously** imputed size to $\mathbf{W_A}$ (25-27) is $\mathbf{W_A}$ (2010), **Tables (2,3)**. Thus our **absolute** Annual wages are:

$$w_i \;=\; \widetilde{w_4} \cdot \frac{\alpha_i \rho_i}{\alpha_4 \rho_4} \frac{(L_i)^{\rho_i - 1}}{(L_4)^{\rho_4 - 1}}\, L_A^{\rho_4 - \rho_i}\ ,\quad \widetilde{w_4} = \frac{W_A\,(2010)\, L}{L_4 + \sum_{i \neq 4}^{M} \frac{w_i}{w_4} L_i}\quad ;\ i = 1, .., M\ ;\ M = 11 \quad (28)$$

By our γ *calibration*[9], hence $\lambda_\mathbf{i} \equiv \frac{\mathbf{L_i}}{\mathbf{L_A}}$, $\sum_{i=1}^{M} \lambda_i = 1$, w_i (28) is *equivalent* to, cf. (22-24), (5):

$$w_i \;=\; \widetilde{w_4} \cdot \frac{\alpha_i \rho_i}{\alpha_4 \rho_4} \frac{\lambda_i^{\rho_i - 1}}{\lambda_4^{\rho_4 - 1}}\ ,\ \widetilde{w_4} = \frac{W_A\,(2010)}{\lambda_4 + \sum_{i \neq 4}^{M} \frac{w_i}{w_4} \lambda_i}\ ,\ \lambda_i \equiv \frac{L_i}{L} = \frac{l_i\, n_i}{l_{15-69}\, n_{15-69}}\ ;\ i = 1, .., M \quad (29)$$

Note that (26-29) give the **same** wages $\mathbf{w_i}$, but CRESH *duality* formulas (25-27) provide economic content and intuition. We saw an illustration of (28-29) in **Table 4**, (Col.6b). For $\mathbf{W_A}$ (2010) = 437552 and, $\mathbf{L_A} = \mathbf{L}$(2010)= $\sum_{i=1}^{M} \mathbf{L_i}$ (2010), with all $\mathbf{L_i}$ (2010) in **Table 2**, (Col.5), the calculation of $\widetilde{\mathbf{w_4}}$ by (28-29) gives, $\widetilde{\mathbf{w_4}}$ (2010) = **410492** ; applying this $\widetilde{\mathbf{w_4}}$ as '*scaling multiplier*' to all wage *ratios*, ($\frac{\mathbf{w_i}}{\mathbf{w_4}}$), **Table 4** (Col.6a), gives CRESH **absolute** (money) **annual wages**, $\mathbf{w_i}$ (2010) (Col.6b) - actual observed (w_i), **data** are in (Col.6c).

Finally, note CRESH formulas (26-29) in **2010** give *higher wages* for w_{60-64}, w_{65-69} than to w_{55-59}, w_{45-49} (despite *lower* substitution *parameters* : ρ_{55-59}, ρ_{45-49}, (Col.5). The influence of much *smaller* Labor *supplies* (scarcity) of L_{60-64}, L_{65-69}, **Table 2** (Col.5), *dominate* (22), (27-28), and explain the *high*, w_{60-64}, w_{65-69}, in both model/data 2010.

CRESH **Age wage profiles**, (26-29), of the age-groups *over time* are complex, but versatile - as will be seen in projected **calendar** *years*, and over entire **cohort** *life* times.

[9]See footnote 8 - where for year 2010 : $\gamma = 4.378$. Using (29), $\gamma's$ to **Tables 5-7** are not needed.

3.2 Disaggregations of Labor Supply - CRESH Subaggregators

The Register based Columns (5-6), **Table 2**, of *age-specific* (labor, wage) *data*, (L_i, w_i), i=1,..,*11*, [making 75 % of GDP (Value Added), $wL/Y = 924.3/1228.1 = 0.75$, cf. **Table 3**] form directly by *Column* (7) *empirical points* outlining a *shape*, seen below in **Fig. 2d**.

The *Age-wage profile* in Columns (6-7) refers to the **complete** Danish **Labor supply** (age 15-69), year 2010 (in *full time equivalents*) : men, women, every occupation, private and public sector, all lengths of schooling, educations, etc.[10]

Standard *Human capital* (Labor quality levels) models posit that earnings (wages) rise with *levels* (years) of *Schooling* (5-7, 9-11, 13-15), or with *Education levels* (High school, College, Graduate school), or with *Occupational* classifications [blue-collar (skilled/craftsmen, unskilled) workers, white-collar (professionals, administrators, clerical) employees].

If available *data* of *age-specific* Labor inputs and wages, $(\mathbf{L_i}, \mathbf{w_i})$, are *disaggregated* (by subscript) : $(\mathbf{L_{iJ}}, \mathbf{w_{iJ}})$, into e.g., **8** *quality* levels (**J**), we may construct **8** CRESH **Subaggregators**, $\mathbf{L_{AJ}} = \mathbf{f_J}(\mathbf{L_{1J}}, \mathbf{L_{2J}}, \ldots, \mathbf{L_{MJ}})$, cf. (10), and hence analogous to (15) get **wage ratios**, $\frac{\mathbf{w_{iJ}}}{\mathbf{w_{jJ}}}$; by analogous **duals** of, (25-27), $W_{AJ}\, L_{AJ} = \sum_{i=1}^{M} w_{iJ} L_{iJ} \equiv w_J L_J \equiv W_J$, the **money wages** (w_{iJ}) of *ages* and *qualities* of Labor input/supply (L_{iJ}) become :

$$w_{iJ}\,(t) = W_{AJ}\ \frac{\partial f_J\,(L_{1J}, L_{2J}, \ldots, L_{MJ})}{\partial L_{iJ}} = W_{AJ}\ \frac{\alpha_{i\,J}\, \rho_{iJ}\, \lambda_{iJ}^{\rho_{iJ}-1}}{\sum_{i=1}^{M} \alpha_{i\,J}\, \rho_{iJ}\, \lambda_{iJ}^{\rho_{iJ}}},\ \ i = 1, .., M,\ \ J = I, II, .., VIII \quad (30)$$

For each **year** (t), disaggregated data $(\mathbf{L_{iJ}}, \mathbf{w_{iJ}})$ can for each **level** (**J**) be organized by *age* (i) as in **Table 2**, Col.(5-6), and the analogous CRESH wage formulas, (29), for each level (**J**) can be implemented for $w_{iJ}\,(t)$, (30), as in **Table 4**, Col.(6a, 6b, 6c), (11,13,15).

Estimating *different* **Age-wage profiles**, $\mathbf{w_{iJ}}$, i=1,..,M, (30), corresponding to *each* school level (**J**), Hanoch (1967, p.315-319) obtained **8** *Age-wage* profiles of essentially *similar* shape, but stacked *vertically* above each other with higher school level (**J**). Although Hanoch (1967) did not formally use Labor subaggregator functions, but *vertical*

[10]The high wages of two *age groups*, (60-64, 65-69), in the Danish Age-wage profiles, **Fig. (2d, 2e)**, **Fig. (12, 13)**, a puzzle, are to some extent, partly due to their high proportions of Public Sector employees with seigniory wage systems (Medical profession in Public Hospitals, other Academics in Government Services (including Universities, Secondary Schools). Lower paid Public Sector employees in Primary School and Hospitals have mostly retired by age 65 in 2010 - as in the Private Sector.
Age-wage profiles on *Disaggregated* Labor data should re-establish global concavity of age-wage profiles.

shifting[11]- on *disaggregated* data - of age-wage profiles by a *specific level* (exogenous) variable is seen as an extension of parametric CRESH age-wage formula (27), where W_A (26) is an *exogenous* vertical shift variable of all (*one* quality level) wages, w_i, i=1,..,M. Thus all *disaggregated* CRESH *Age-wage profiles*, $w_{iJ}\,(t)$, (30), are for any year (t) stacked vertically[12] through their quality **level** *Average wage*, W_{AJ}, cf. (25).

To see clearly how **Subaggregators**, $L_{AJ} = f_J(L_{1J}, L_{2J}, \ldots, L_{MJ})$, work in (30) without attention to Total Labor supply numbers ($L_{A\,J}$), we may recall that CRESH function, (7), is *homogeneous* of degree **zero** : $F(L_A, L_1, L_2, \ldots, L_M) = F(1, \lambda_1, \lambda_2, \ldots, \lambda_M)$, and that *Aggregator*, $L_A = f\,(L_1, L_2, ..., L_M)$, (10), *Subaggregators*, are *homogeneous* of degree **one** - implying that all their *partial derivatives* : $\frac{\partial f_J(L_{1J},L_{2J},\ldots,L_{MJ})}{\partial L_{iJ}}$, are *homogeneous* functions of degree **zero**, such that as stated in (12), (30), we have partial derivatives :

$$\frac{\partial f_J\,(L_{1J}, L_{2J}, \ldots, L_{MJ})}{\partial L_{iJ}} = \frac{\partial f_J\,(\lambda_{1J}, \lambda_{2J}, \ldots, \lambda_{MJ})}{\partial \lambda_{iJ}} = \frac{\alpha_{i\,J}\,\rho_{iJ}\,\lambda_{iJ}^{\rho_{iJ}\,-1}}{\sum_{i=1}^{M} \alpha_{i\,J}\,\rho_{iJ}\,\lambda_{iJ}^{\rho_{iJ}}}\ ,\ i = 1, ., M,\ J = I, ., \mathit{VIII} \quad (31)$$

By CES (9), the derivatives (31) are much simplified, as denominator above drops out:

$$\forall\, i:\ \rho_{iJ} = \rho_J\ ,\ \ \frac{\partial f_J\,(\lambda_{1J}, \lambda_{2J}, \ldots, \lambda_{MJ})}{\partial \lambda_{iJ}} = \alpha_{i\,J}\,\lambda_{iJ}^{\rho_J\,-1},\ \ i = 1, ., M,\ J = I, ., \mathit{VIII}\ ;\ \sum_{i=1}^{M} \alpha_{i\,J}\,\lambda_{iJ}^{\rho_J} = 1 \quad (32)$$

Only *derivatives* (31-32) of *Aggregator* functions, $L_{AJ} = f_J$, are used in *imputing wages* w_{iJ}, (30), to the *age-groups* (i) of Total Labor supply, $L_{AJ} = \sum_{i=1}^{M} L_{i\,J}$.

The *derivatives* (27), (31-32) are **not** *marginal products* (output) of L_i in age group (i), but *marginal contributions* of L_i to L_A by the *Aggregator* function, $\frac{\partial L_A}{\partial L_i} = \frac{\partial f}{\partial L_i}$, cf. (12); marginal contributions, $\frac{\partial L_A}{\partial L_i}$, do *not* depend on (invariant to) the *absolute* sizes of (L_A, L_i), but only upon the *size* of λ_i, in CES, (32) - and upon **all** λ_i with CRESH, (31).

As in (15) efficient utilization of Labor supplies - within $L_{A\,J} = \sum_{i=1}^{M} L_{i\,J},\ J = I, II$ - requires that the ratio (relative) of age-wages were equated to the ratio (relative) of their marginal contribution : $\frac{\mathbf{w_{i\,J}}}{\mathbf{w_{j\,J}}} = \frac{\partial f_J(\lambda_{1J},\lambda_{2J},\ldots,\lambda_{MJ})\,/\,\partial \lambda_{iJ}}{\partial f_J(\lambda_{1J},\lambda_{2J},\ldots,\lambda_{MJ})\,/\,\partial \lambda_{jJ}}$, $i \neq j$. With data and the accounting identities, $W_{AJ}\,L_{AJ} = \sum_{i=1}^{M} w_{iJ} L_{iJ} \equiv W_J$, $J = I, ., \mathit{VIII}$, we have the Average wages : $W_{AJ} = W_J / L_{AJ}$, $J = I, ., \mathit{VIII}$. Thus, by (31) and, W_{AJ} , we have also the *absolute money* wages ($w_{i\,J}$) for *all Age groups* (M) in *all* the *Labor categories* (qualities), (VIII) :

$$\mathbf{w_{i\,J}\,(t)}\ =\ W_{AJ}\ \frac{\partial f_J\,(\lambda_{1J}, \lambda_{2J}, \ldots, \lambda_{MJ})}{\partial \lambda_{iJ}}\ =\ W_{AJ}\ \frac{\alpha_{i\,J}\,\rho_{iJ}\,\lambda_{iJ}^{\rho_{iJ}\,-1}}{\sum_{i=1}^{M} \alpha_{i\,J}\,\rho_{iJ}\,\lambda_{iJ}^{\rho_{iJ}}}\ ,\ i = 1, ., M,\ J = I, ., \mathit{VIII} \quad (33)$$

which are the CRESH **Calendar year** (t) **Age-wage profiles**, (30), , restated in λ_{iJ}.

[11]The disaggregated age-wage profiles not only shift vertically, but they may also twist/rotate.

[12]Age-earnings (wage) profiles from education have a long economic history, Blaug (1967, p.337).

Thus analytic **wage structure** description for *different* Labor *qualities* (education levels) require the analytic tools of CRESH **Subaggregator**, $L_{AJ} = f_J(L_{1J}, L_{2J}, \ldots, L_{MJ})$, *derivatives* (31) as used in (33)[13]. See hereto the *Canonical Model* in **Appendix C**. Finally, we note that changes in *wage structure* (distributions) can be *analyzed* in calendar years (section 4) by the apparatus of CRESH Labor *Aggregators* [not using solely total Labor supplies, but only *age* distributions, λ_{iJ}, $i = 1, .., M$] - *without* production functions.

3.3 Outputs and Multi-factor CRESH Production Functions

For a long time, the scope of *Macro* (Y) models has been enlarged by increasing the number of *primary factors.* But here a problem has also existed for years, viz. that with more than two factors the multi-factor CES function has the *same constant* substitution *elasticity* (σ) between any and all factors - *severe* restriction that we *removed* by CRESH Labor *aggregator*, $F(L_A, L_1, L_2, \ldots, L_M) = 0$, (7), (10), and the *Sub-aggregators* above.

The CRESH functional form can also be used to CRESH implicit *production* functions:

$$G(Y, X_I, X_{II}, .., X_V) = G\left(Y, L_I, L_{II}, K_{III}, K_{IV}, K_V\right) \;=\; \gamma \sum_{J=I}^{V} \alpha_J \left[\frac{X_J}{Y}\right]^{\rho_J} - 1 \;=\; 0 \quad (34)$$

$$\gamma > 0; \;\; \forall_J : \; \alpha_J > 0, \;\; \sum_{J=I}^{V} \alpha_J = 1; \;\; \forall_J : \;\; 0 < \; \rho_J \leq 1 \;\; or \;\; \rho_J < 0 \quad (35)$$

where the *parameters* (35) again preserve the important *global* regularity properties. As in (10), a **unique** implicit **production function**, $\mathbf{Y} = \mathbf{g}\left(\mathbf{X_I}, \mathbf{X_{II}}, ., \mathbf{X_V}\right)$ exists and g

$$\forall\, X_J > 0 \; : \;\; Y = g\left(X_I, X_{II}, .., X_V\right) \; > 0 \; ; \;\; \frac{\partial g}{\partial X_J} > 0 \, , \; \frac{\partial^2 g}{\partial X_J^2} < 0 \; ; \; Y \; = \; \sum_{J=1}^{V} \frac{\partial g}{\partial X_J} X_J \quad (36)$$

having all the globally regularity properties as the Labor Aggregator, $L_A = f$, (10). All expressions and illustrations of the *Substitution* elasticities and the *Complementarity* elasticities in **Appendix A-B** carry over to (34-36).

Old problems with *different* substitution elasticities between **two** Labor categories, L_I, L_{II}, and various **nonlabor** inputs such as services of *Capital* goods[14], (34), can be

[13] Labor Aggregator derivatives, (27), (30), (33), are analogous to '*Inverse* factor (consumer) demand functions' by derivatives of production (utility) functions; first-order and second-order derivatives define *complementarity* elasticities, c_{ij}, (56), (83), giving *wage* elasticities, (63), (86-88), w.r.t Labor *supplies.*

[14] See Berndt and Cristensen (1974, p.391-92) ; cf. skill-biased technological change in footnote 25.

resolved with proper *Macro* wage numbers assigned to $\mathbf{W_{AJ}}$, $J = I, II$ - and subsequently used for the *Age-wage profiles* of the two Labor Subaggregates, $\mathbf{w_{i\,J}}$, in (33-34).

Analogously to (27), Macro money *wages* $\mathbf{W_{AJ}}$, $J = I, II$, are simply derived from **CRESH** macro *production function*, (34-36) (single output, Y) and the *output price* (P):

$$W_{AI} = P \cdot \frac{\partial g\left(L_I, L_{II}, K_{III}, K_{IV}, K_V\right)}{\partial L_I} = P \cdot \frac{\alpha_I \rho_I \left(L_I/Y\right)^{\rho_I - 1}}{\sum_{J=I}^{V} \alpha_J \rho_J \left(X_J/Y\right)^{\rho_J}} \tag{37}$$

$$W_{AII} = P \cdot \frac{\partial g\left(L_I, L_{II}, K_{III}, K_{IV}, K_V\right)}{\partial L_{II}} = P \cdot \frac{\alpha_{II} \rho_{II} \left(L_{II}/Y\right)^{\rho_{II} - 1}}{\sum_{J=I}^{V} \alpha_J \rho_J \left(X_J/Y\right)^{\rho_J}} \tag{38}$$

Depending on the evolution (time series) of factor productivities (unit requirements), $(L_I/Y, L_{II}/Y, K_{III}/Y, K_{IV}/Y, K_V/Y)$, the sizes of the two **Macro wages** (37), (38), are changing, which **shift** the **Calendar year** *Subaggregate* (Micro) *Age-wage* **profiles**, (33). Shifting of (33) by W_{AJ}, (37-38), does not alter the *shape* of (33) and its *relative* wages.

3.3.1. *Inverse Labor demands - Wage functions - Age-wage profiles, and Empiric methods*

Standard labor demand analyses have estimated various *explicit* production functions, $Y = G\left(L_I, L_{II}, L_{III}, L_{IV}, L_V, K_I, K_{II}\right)$, as e.g., Trans-Log, Freeman (1979), cf. Introduction, Hamermesh & Grant (1979, p.538; 1981, p.357), or Generalized Leontief production function, Borjas (1986, p.59), to obtain relevant Labor demand functions, complementarity elasticities and partial wage elasticities - survey in Hamermesh (1993). In Multi-factor *Production functions*, many *classifications* into Labor & wage sub-groups were used: various occupations, educations (length of schooling), gender (male, female), age (young, middle age, old). However, we are *not* using production functions at all ; we have no proper data for capital inputs (quantities or their factor prices). Instead, we have for our purposes a complete data set of Danish Labor supplies and wages, seen in sections 2.2, 3.1. Hence we used (constructed) and estimated (calibrated) the CRESH *Labor Aggregator function*, $L_A = f\left(L_1, L_2, ..., L_M\right)$, (7-8), (10), and accordingly here get its *Inverse Labor demand* system as, $w_i = W_A \cdot \frac{\partial f}{\partial L_i}(L_1, .., L_M)$, $i = 1, 2, ., M$, cf. (92), or as *wage* functions also called *Age-Annual Wage profiles*, which in *explicit parametric* CRESH *form* is stated in the *equation*, (27). The CRESH *Age-Annual wage* formula (27) is used in sections 4-5 to perform analytic '**controlled experiments**' of *Demographic impacts* upon Calender year wages and Cohort life cycle wages. In these scenarios, the benchmark value of W_A is $\mathbf{W_A}$ (2010) - and hence (27) becomes *'operative'* as (28-29).

4 Demographics, Labor supplies, and Calendar wages

4.1 Projected labor age groups, relative wages, annual wages

Danish Population sizes, $\mathbf{N_i(t)}$, for the **11** *age-groups* (i) of *working life* (15-69) - obtained from United Nations (2015) source, cf. **Table 1** - are seen in **Tables (5a, 5b)**, Col.1. Danish Labor Supplies (full-time workers), $\mathbf{L_i(t)}$, (39), in *age-groups* (i) - calculated by $\mathbf{N_i(t)}$, (4), and *Labor Participation* rates, $\mathbf{l_i}$ (2010), **Table 2** - are **Tables (5a,5b)**, Col.2.,

$$L_i(t) = l_i\,(2010) \cdot N_i(t)\,,\; t \;=\; 2020,\, 2030,\, 2040,\, 2050,\, 2070,\, 2090\,;\; L(t) = \sum_{i=1}^{M} L_i(t) \quad (39)$$

e.g., $L_{15-19}(2020) = 0.0855 \cdot 338740 = 28962$, $L_{35-39}(2030) = 0.7468 \cdot 412710 = 308212$. The Participation rates as $\mathbf{l_i}$ (2010) are held **constant** through the whole *demographic projection* **period** (2020-2090), and for **all** (*Medium, Low, High*) demographic variants. In the *three* Fertility variants described in **Table 1a**, the *Fertility* change *commences* in **2015**, cf. **Fig.1**. This implies that *Labor Supplies*, $L_i(t)$, starting $i = 15$ - are *equal* for all fertility variants **until** 2030 ; hence 2035 is the *first* five year period in which labor supplies, $L_i(t)$ *differ* across the *three* fertility variants. Hence we report population, labor supplies for 2020 and 2030 separately in **Table 5a**, since these are common to all variants. But age-specific *wages* $w_i(t)$ are *not* constant for 2020, 2030, as they have different $L_i(t)$. **Table 5b** extends **Table 5a** for 2040 to 2090 for the *three* fertility variants. In **Tables (5a, 5b)**, last Col. are shown in all years/variants the *Age wage profile* of 2010, $\mathbf{w_i}$ (**2010**). The **relative** age-group *wages*, $w_i(t)/w_4(t)$ - calculated by inserting $L_i(t)$ and total $L(t)$ from (39) into CRESH, (22), with $L(t) = L_A$ - are exhibited in **Tables (5a, 5b)**, Col.3.

$$\frac{w_i(t)}{w_4(t)} = \frac{\alpha_i \rho_i}{\alpha_4 \rho_4} \frac{L_i(t)^{\rho_i - 1}}{L_4(t)^{\rho_4 - 1}}\, L(t)^{\rho_4 - \rho_i}\,, \quad t \;=\; 2020,\, 2030,\, 2040,\, 2050,\, 2070,\, 2090 \qquad (40)$$

The values for $w_i(t)/w_4(t)$ in **Table 5b** differ for each variant, as $\mathbf{L_i(t)}$ (column 2) differ. The conforming *absolute/money age-group wages*, $w_i(t)$ - by multiplying (40) with *money* **wage**, $\widetilde{w_4}(t)$, with $W_A(2010) = 437552$, cf. (28), (84) - are in **Tables (5a, 5b)**, Col.4 :

$$w_i(t) \;=\; \widetilde{w_4}(t) \cdot \frac{\alpha_i \rho_i}{\alpha_4 \rho_4} \frac{L_i(t)^{\rho_i - 1}}{L_4(t)^{\rho_4 - 1}}\, L(t)^{\rho_4 - \rho_i},\; i = 1, .., 11\,;\;\; \widetilde{w_4}(t) = \frac{W_A(2010)\, L(t)}{L_4(t) + \sum_{i \neq 4}^{11} \frac{w_i(t)}{w_4(t)} L_i(t)} \quad (41)$$

Thus **Tables (5a, 5b)** present 'comparative' *age-group annual wages* in *two forms* : directly as **ratio**: $\frac{w_i(t)}{w_4(t)}$, (40), and on *absolute* income scale as : money **wage**, $w_i(t)$, (41).

Both these two forms (40-41) are necessary for calculating and understanding the influence of demographic *projections* $N_i(t)$ via $L_i(t)$, (39), (4), upon *any* and **all** (11) '*age* group (i) *annual wages*', $w_i(t)$, (41), (29) - **Age-wage profile** - in **Calendar years** (t), **Tables (5a, 5b)**.[15] Later *translated* $w_i(t)$ of (41) are used in *summing* annual wages of *Cohorts*, (43-44), **Table 7**, during their whole *working life* (employment years, $i : 15-69$).

Table 5a. Population and Employment projections, Relative wages, Annual wages - Denmark						
		All Variants				
	Age (*i*)	N_i	L_i	w_i/w_4	w_i	w_i (2010)
2020	15-19	338740	28970	0.4617	188933	187005
	20-24	371920	124262	0.6529	267205	273220
	25-29	398400	214793	0.8141	333172	358262
	30-34	353410	248380	1.0000	409251	410668
	35-39	316830	236594	1.1949	489015	449679
	40-44	359770	269781	1.2309	503731	471118
	45-49	377910	282547	1.1996	490917	472491
	50-54	424970	312567	1.0889	445644	471381
	55-59	383500	265371	1.1006	450428	461729
	60-64	345940	119216	1.1868	485695	478708
	65-69	310130	22380	1.1814	483504	483248
	Total	3981520	2124861		437552	437552
2030	15-19	309940	26507	0.4858	191356	187005
	20-24	359050	119962	0.6797	267744	273220
	25-29	365930	197287	0.8632	340041	358262
	30-34	394580	277315	1.0000	393930	410668
	35-39	412710	308193	1.1110	437674	449679
	40-44	361800	271303	1.2685	499693	471118
	45-49	320320	239490	1.3465	530442	472491
	50-54	356350	262097	1.2076	475705	471381
	55-59	368240	254811	1.1515	453630	461729
	60-64	407810	140537	1.1871	467615	478708
	65-69	358960	25904	1.1860	467214	483248
	Total	4015690	2123406		437552	437552

Source: See **Table 5b.**

[15]For $\mathbf{t = 2020}$: N_{15-19} are born in the (*Generation* from) *calendar years* (t): 2001-2005 ; similarly, N_{20-24} born 1996-2000; N_{25-29} born 1991-1995; N_{30-34} born 1986-1990; N_{35-39} born 1981-1985; N_{40-44} born 1976-1980; N_{45-49} born 1971-1975; N_{50-54} born 1966-1970; N_{55-59} born 1961-1965; N_{60-64} born 1956-1960; N_{65-69} born 1951-1955. For $\mathbf{t = 2030}$: N_{15-19} are born in *calendar years* (t): 2011-2015, and, N_{20-24} born 2006-2010; N_{25-29} born 2001-2005; N_{30-34} born 1996-2000; , ; N_{65-69} born 1961-1965.
For $\mathbf{t = 2040}$: N_{15-19} are born in *calendar years* (t): 2021-2025; , ; N_{65-69} born 1971-1975.
For $\mathbf{t = 2050}$: N_{15-19} are born in *calendar years* (t): 2031-2035; , ; N_{65-69} born 1981-1985.
For $\mathbf{t = 2070}$: N_{15-19} are born in *calendar years* (t): 2051-2055; , ; N_{65-69} born 2001-2005.
For $\mathbf{t = 2090}$: N_{15-19} are born in *calendar years* (t): 2071-2075; , ; N_{65-69} born 2021-2025.
The *pure* impact of increased *Life Expectancy*, cf. **Table 1a**, upon *Population* numbers N_i may noted by comparing for t=2020, t=2030, the *corresponding sizes* of the *age group* of *same* birth years - e.g., N_{25-29} (2030) = 365930 > N_{15-19} (2020) = 338740 ; N_{35-39} (2030) = 412710 > N_{25-29} (2020) = 398400.
The so-called *Millennial* Generation (**Y**), born (**1981-1995**), is seen, $\mathbf{t = 2020}$ as : $N_{35-39} + N_{30-34} + N_{25-29}$. The *Generation* (**Z**), born (1996-2010), is seen above, $\mathbf{t = 2030}$ as : $N_{30-34} + N_{25-29} + N_{20-24}$.

Table 5b. Population and Employment projections (Labor supplies), Relative annual wages (Age wage profiles), Annual wages - Denmark

		Medium Variant				Low Variant				High Variant				
	Age (i)	N_i	L_i	w_i/w_4	w_i	N_i	L_i	w_i/w_4	w_i	N_i	L_i	w_i/w_4	w_i	w_i (2010)
2040	15-19	345610	29558	0.4708	188390	272370	23294	0.4943	196878	418860	35823	0.4526	181947	187005
	20-24	336720	112502	0.6820	272882	293580	98088	0.7017	279471	379860	126915	0.6651	267355	273220
	25-29	337320	181862	0.8759	350480	337320	181862	0.8759	348891	337320	181862	0.8759	352103	358262
	30-34	381930	268424	1.0000	400119	381930	268424	1.0000	398306	381930	268424	1.0000	401972	410668
	35-39	380600	284214	1.1361	454562	380600	284214	1.1350	452058	380600	284214	1.1372	457112	449679
	40-44	403130	302296	1.1892	475811	403130	302296	1.1868	472725	403130	302296	1.1915	478945	471118
	45-49	415960	310995	1.1693	467867	415960	310995	1.1670	464833	415960	310995	1.1716	470949	472491
	50-54	359430	264362	1.1913	476666	359430	264362	1.1901	474040	359430	264362	1.1925	479340	471381
	55-59	314110	217355	1.1961	478566	314110	217355	1.1961	476396	314110	217355	1.1961	480782	461729
	60-64	345740	119147	1.2154	486304	345740	119147	1.2166	484575	345740	119147	1.2142	488081	478708
	65-69	349650	25232	1.1811	472581	349650	25232	1.1823	470901	349650	25232	1.1800	474308	483248
	Total	3970200	2115947		437552	3853820	2095269		437552	4086590	2136625		437552	437552
2050	15-19	358920	30696	0.4578	187440	265200	22681	0.4705	196405	452750	38721	0.4498	181473	187005
	20-24	373010	124627	0.6545	268022	278730	93127	0.6712	280211	467300	156130	0.6441	259847	273220
	25-29	373030	201115	0.8348	341834	299900	161688	0.8578	358107	446170	240548	0.8184	330170	358262
	30-34	359790	252864	1.0000	409479	316730	222601	1.0000	417453	402850	283127	1.0000	403417	410668
	35-39	352260	263051	1.1541	472578	352260	263051	1.1051	461321	352260	263051	1.1998	484008	449679
	40-44	390850	293087	1.1927	488388	390850	293087	1.1362	474304	390850	293087	1.2460	502659	471118
	45-49	384650	287586	1.2009	491733	384650	287586	1.1440	477552	384650	287586	1.2545	506101	472491
	50-54	401150	295048	1.1229	459807	401150	295048	1.0752	448854	401150	295048	1.1673	470928	471381
	55-59	409140	283113	1.0853	444392	409140	283113	1.0445	436048	409140	283113	1.1227	452915	461729
	60-64	351200	121029	1.1869	485996	351200	121029	1.1482	479335	351200	121029	1.2218	492895	478708
	65-69	302090	21800	1.1913	487822	302090	21800	1.1526	481136	302090	21800	1.2264	494747	483248
	Total	4056090	2174016		437552	3751900	2064811		437552	4360410	2283240		437552	437552
2070	15-19	355310	30387	0.4675	187910	224120	19168	0.4813	196251	515840	44117	0.4556	182023	187005
	20-24	357800	119545	0.6727	270394	246230	82268	0.6805	277519	485980	162371	0.6644	265435	273220
	25-29	368850	198861	0.8541	343271	271430	146339	0.8618	351421	472460	254722	0.8468	338317	358262
	30-34	383870	269788	1.0000	401926	291060	204560	1.0000	407792	477950	335908	1.0000	399510	410668
	35-39	396630	296185	1.1226	451193	303220	226431	1.1277	459879	490150	366021	1.1204	447622	449679
	40-44	401890	301366	1.2001	482346	308040	230990	1.2125	494466	495750	371749	1.1939	476972	471118
	45-49	390560	292005	1.2159	488720	317960	237725	1.1921	486123	463160	346285	1.2337	492890	472491
	50-54	367500	270298	1.1862	476753	324960	239010	1.1242	458453	410040	301586	1.2333	492711	471381
	55-59	352640	244016	1.1570	465032	352640	244016	1.0648	434226	352640	244016	1.2357	493656	461729
	60-64	383430	132136	1.1887	477769	383430	132136	1.1159	455043	383430	132136	1.2481	498625	478708
	65-69	369080	26634	1.1666	468895	369080	26634	1.0951	446591	369080	26634	1.2249	489364	483248
	Total	4127560	2181221		437552	3392170	1789277		437552	4916480	2585545		437552	437552
2090	15-19	356750	30511	0.4678	187931	197880	16923	0.4733	195744	572910	48997	0.4643	182664	187005
	20-24	370270	123711	0.6691	268794	214020	71506	0.6714	277672	575620	192321	0.6683	262964	273220
	25-29	382930	206453	0.8472	340305	228820	123366	0.8574	354595	581740	313639	0.8404	330648	358262
	30-34	387900	272620	1.0000	401696	241260	169560	1.0000	413565	573670	403181	1.0000	393454	410668
	35-39	384620	287216	1.1419	458701	253740	189481	1.1280	466488	544760	406801	1.1517	453129	449679
	40-44	380400	285251	1.2415	498691	269200	201865	1.1904	492318	508140	381040	1.2835	505013	471118
	45-49	382680	286113	1.2363	496622	285770	213658	1.1540	477273	485760	363182	1.3112	515911	472491
	50-54	389970	286825	1.1639	467530	297960	219151	1.0841	448345	483250	355433	1.2383	487208	471381
	55-59	396800	274574	1.1203	450011	304750	210878	1.0516	434898	488960	338346	1.1833	465578	461729
	60-64	396730	136719	1.1824	474956	305110	105145	1.1205	463405	488350	168292	1.2374	486862	478708
	65-69	379450	27383	1.1619	466731	309620	22343	1.0881	450019	449280	32422	1.2255	482162	483248
	Total	4208500	2217376		437552	2908130	1543876		437552	5752440	3003654		437552	437552

Source: N_i, United Nations (2015), cf. **Tables (1, 2)**; $L_i = l_i\,(2010) \cdot N_i(t)$, (4), (39); $\frac{w_i(t)}{w_4(t)}$, (40); $w_i(t)$, (41).

Table 5c. Labor supplies and Annual wages of Younger (30-34), Middle aged (45-49), and Older (55-59) workers in Calendar years : 2020, 2030, 2040, 2050, 2070, 2090 – Three Fertility (M, L, H) variants.							
Year		L_{30-34}	w_{30-34}	L_{45-49}	w_{45-49}	L_{55-59}	w_{55-59}
2020	-	248380	409251	282547	490917	265371	450428
2030	-	277315	393930	239490	530442	254811	453630
2040	M	268424	400119	310995	467867	217355	478566
	L	268424	398306	310995	464833	217355	476396
	H	268424	457112	310995	470949	217355	480782
2050	M	252864	409479	287586	491733	283113	444392
	L	222601	417453	287586	477552	283113	436048
	H	283127	403417	287586	506101	283113	452915
2070	M	269788	401926	292005	488720	244016	465032
	L	204560	407792	237725	486123	244016	434226
	H	395908	399510	346285	492890	244016	493656
2090	M	272620	401696	286113	496622	274574	450011
	L	169560	413565	213658	477273	210878	434898
	H	403181	393454	363182	515911	338346	465578

Source: **Tables (5a,5b)**, rows, **age (i)** : 30-34, 45-49, 55-59.

Based on **Table 5c** we show in **Figures (2a-2c)** the annual wages $\mathbf{w_i(t)}$ for *younger* (30-34 years), *middle aged* (45-49 years) and *older* (55-59 years) workers in each of the *calendar years* 2020, 2030, 2040, 2050, 2070, 2090, assuming *three* different *fertility* levels.

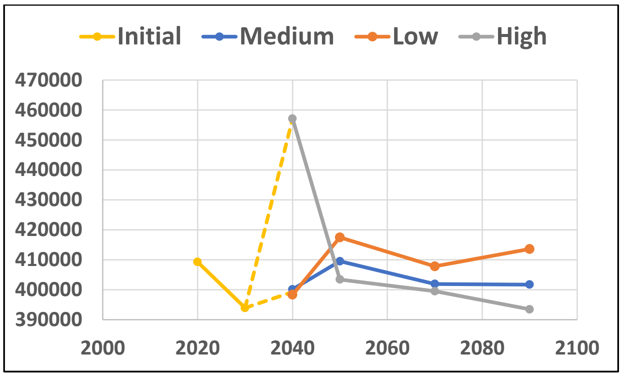
First, it is possible from **Table 5c**, **Fig. (2a-2c)**, to get an impression of the importance for the *annual wages* of belonging to a *small* (respectively big) *Birth group* (*Generation*), **Tables (5a, 5b)** and hence their subsequent *Labor supplies* in **Table 5c**. The clearest example of this is found by looking at w_{30-34}, **Fig. 2a**, of the 30-34 years old in 2020, L_{30-34}, born as a very *small* generation in *1986-1990*[16], next at the *highest* w_{45-49}, **Fig. 2b**, of the 45-49 years old in **2030**, L_{45-49}, born as the even *smaller* generation in *1981-1985*[17], and finally at the high w_{55-59}, **Fig. 2c**, of the 55-59 years old in 2040, L_{55-59}, born also in same years, *1981-1985*, (Generation, Y). In all cases, the wage-supply response to belonging to **small** generations (early *Millennial*) is a **high** annual wage.

Secondly, **Fig.2a** shows how *Low* fertility permanently from 2050 creates a scarcity of workers 30-34 years old, resulting in higher wages w_{30-34} from 2050. Higher fertility does *not* help wages of *younger members* (L_{30-34}) of *Labor Supply*, $L(t)$; **Fig. (2b-2c)**, 2050-2090, show clearly how *Higher* fertility *raise* wages of *middle-aged* and *older* workers.

[16]Belonging to (Y), N_{30-34}, t=2020, in footnote 10.

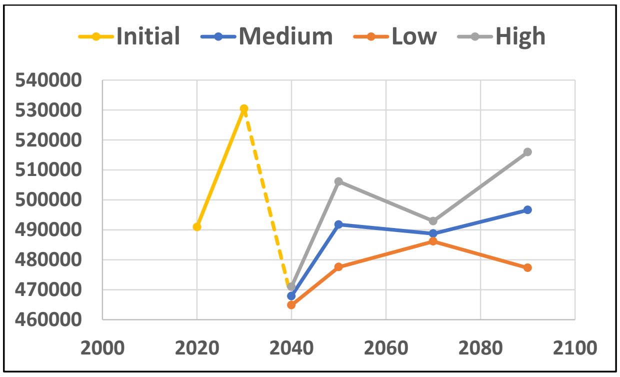
[17]Belonging to (Y), N_{45-49}, t=2030, in footnote 10.

Fig. 2a. *Annual wages for younger workers* $\mathbf{w_{30-34}}(\mathbf{t})$ *in* **calendar** *years (t): 2020, ,2090 - in three variants : Medium, Low, High fertility.*



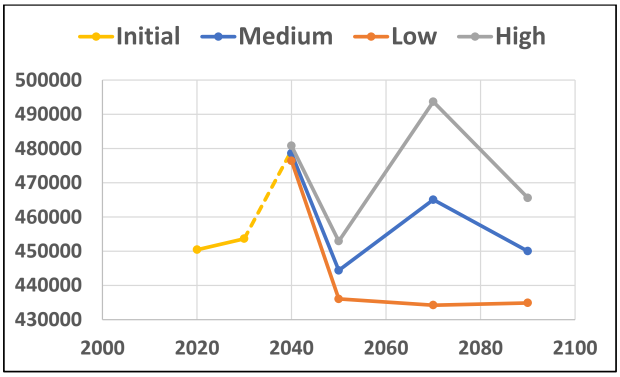
<u>Source</u>: The six numbers of $\mathbf{w_{30-34}}(\mathbf{t})$ - for each variant - are seen in **Table 5c.**

Fig. 2b. *Annual wages for the age group* $\mathbf{w_{45-49}}(\mathbf{t})$ *in* **calendar** *years (t): 2020, ,2090 - in three variants : Medium, Low, High fertility.*



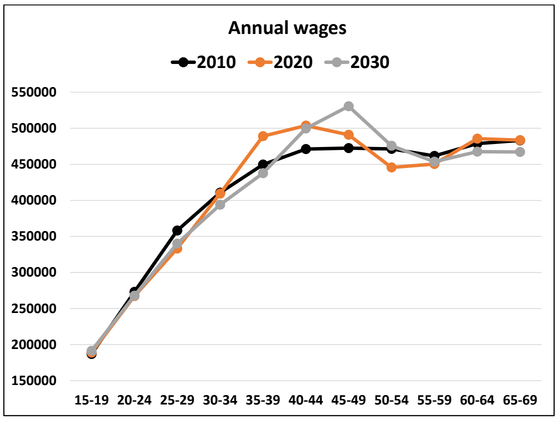
<u>Source</u>: The six numbers of $\mathbf{w_{45-49}}(\mathbf{t})$ - for each variant - are seen in **Table 5c.**

Fig. 2c. *Annual wages for older workers* $\mathbf{w_{55-59}}(\mathbf{t})$ *in* **calendar** *years (t): 2020, ,2090 - in three variants : Medium, Low, High fertility.*



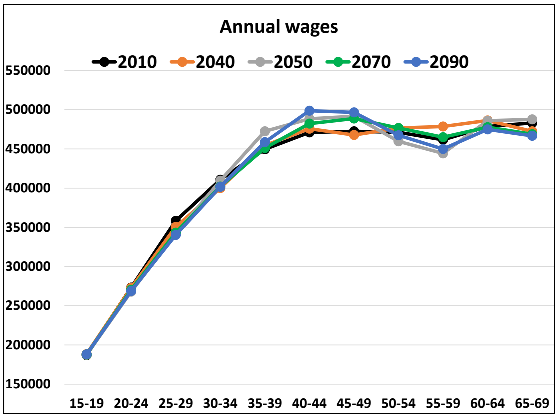
<u>Source</u>: The six numbers of $\mathbf{w_{55-59}}(\mathbf{t})$ - for each variant - are seen in **Table 5c.**

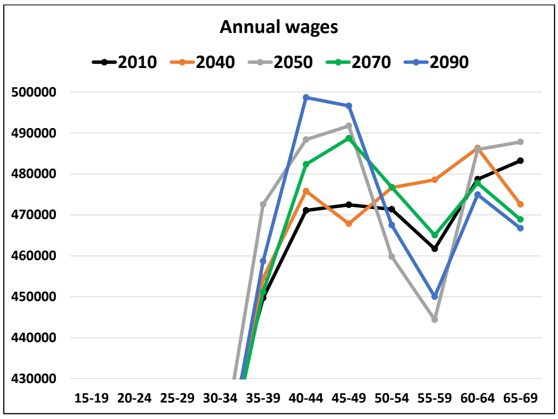
Fig. 2d. Age-wage profiles, $w_i\,(t)$, i= 1, 2, , M, age group : $i = 1 =$ 15-19, $i = 11 =$ 65-69 - for the **calendar** years, $t =$ 2010, 2020, 2030.



Source: Sizes of annual wages, $\mathbf{w_i\,(t)}$, (41), (29), $t =$ 2010, 2020, 2030, see **Table 5a.**

Fig. 2e. Age-wage profiles, $w_i\,(t)$, i= 1, 2, , M, $i = 1 =$ 15-19, $i = 11 =$ 65-69 - for the **calendar** years, $t =$ 2010, 2040, 2050, 2070, 2090 : **Medium** fertility.





Source: Annual wages, $\mathbf{w_i\,(t)}$, (41), (29), $t =$ 2010, 2040, 2050, 2070, 2090, see **Table 5b**.

Whereas **Fig. (2a-2c)** focused on the *fertility* variants and showing their wage impacts upon particular age groups at selected time points, we exhibit in **Fig. (2d)** annual *wages* $\mathbf{w_i(t)}$ of *all* (overlapping) 11 age groups (i) - *Age wage profile* - for 3 **calendar** years (t). The observed *Age wage profile* of 2010, $\mathbf{w_i\,(2010)}$, **Table 2** (col.6), is seen (black) in **Fig. (2d)** - and in **Fig. (2e)** (two panels), where $\mathbf{w_i(t)}$ is shown for 5 **calendar** years (t).

Apart from the last two age groups (60-64, 65-69), the *shape* of the *calendar* year (t) Danish *Age-wage profiles*, **Fig. (2d, 2e)**, are *qualitatively* the same (**concave**) - a shape of Age-wage profiles that we shall later see extended to **Cohorts** (T) in **Fig. (12, 13)**.

4.2 Age groups, LFP, Support ratios, Annual wages: 2020-2090

Table 6 provides *demographic summary* variables and *wage incomes* of national accounts. *Row* 1 gives the total *working* age *population size*, N_{15-69}, which corresponds to the **Totals** of N_i in **Tables 5a, 5b**, (column 2). *Row* 2-4 give the *population* sizes, N_{0-14}, N_{70+}, N, from which the Danish *Dependency ratios*, (1) in **Table 1b** were derived.

Row 5 gives the Labor Force (*Labor Supply*), L_{15-69}, the **Totals** of L_i in **Tables 5a, 5b**, (column 3), where the L_i were generated by (39). *Row* 6 (ratio of row 5/row 1) give *sizes* of the **macro** (endogenous) *Labor Force Participation* rate (l_{15-69}), (**LFP**), (2).

The Danish macro **LFP** (l_{15-69}), (2), (39), for population projection *period 2010-2090* are shown in **Fig. 3** below. In the *High* fertility scenario, the **LFP** (l_{15-69}) is close to *stationary* in the 50 years from 2040 to 2090 as the population N_{15-69} grows at the same rate as L_{15-69}. In the *Low* fertility scenario, we find some changes from 2030 to 2050, as the Population 15-69 years old, (N_{15-69}), falls more than the Labor Supply, (L_{15-69}), while both magnitudes fall at the same rate from 2050.

Fig. 4 illustrates (based on **Table 6**), the *dramatic long-run* consequences regarding the *composition* of the *population* by age groups *outside* the labor force. For the 0-14 years old (N_{0-14}), the **range** is between 10 and 20 percent of the population (N) for the *Low*, respectively the *High* fertility case. An even bigger *range* is found for the *share* of the population 70 years and older, N_{70+}. Until 2070, the *upper* part of **Fig. 4** shows a low fertility '*dividend*'. The *shift* in the *last* 20 years (2070-2090) is due to large increase in the *dependency rate* for the 70+ group, $\mathbf{d_o}$, (1), cf. **Fig. 1**, in the *Low* fertility projection.

Row 7 gives the **Support ratios**, L/N, (3-5), for the period 2020-2090. The *Support ratio* (3) for 2010, **Table 3a**, was given in (6). While Support ratios for Denmark are widely available (World Bank and OECD, for example[18]), our calculations in **Tables (5a, 5b)** show how $L = L_{15-69}$ in the *Support ratio*, **Table 6** (row 5), are obtained as the *Total* of the *same* 11 *age-specific* Labor *supplies* $L_i(t)$ that are used for calculating the CRESH *relative* wages, (40), CRESH *absolute* wages, (41), seen in **Tables (5a, 5b)**.

Danish **Support ratios** for the whole population, $\mathbf{L/N}$, (3), (6), for 2010-2090 are shown in **Fig. 5**. Compared with the **LFP** (l_{15-69}) in **Fig. 3**, the only difference is - as

[18]See, http://data.worldbank.org/ ; https://data.oecd.org/society.htm.

expected - found in the *terminal* year **2090**, where the *Support* ratio (L/N) in the *High* fertility case is *higher* than in the *Low* fertility case, as a consequence of the increasing *share* of 0-14 years old, N_{0-14}/N, cf. **Fig. 4**.

Row 8 gives **Total Wage** Income (wL), as *sum* of all the **11** age groups $[\,L_i(t)\cdot w_i(t)\,]$ in **Tables (5a, 5b)**, (columns 3, 5), in *calendar year* (t).

Row 9 gives (as explained for **Table 3a**) similarly, *Total Wage Income* per **capita**, wL/N, decomposed as : $\mathbf{W_A}\cdot$ *Support ratio*, with $\mathbf{W_A} = \mathbf{w}(2010) = 437552$. The $\mathbf{w}\cdot\mathbf{L}(\mathbf{t})/\mathbf{N}(\mathbf{t})$ for 2020-2090 are shown in **Fig. 6**. Over *30 years*, 2020-2050, wage income *per capita* in **Fig. 6** is significantly *lower* - the *higher* the fertility is - as high growth first in the *younger* parts (L_i) of the Labor supplies, $L(t)$, due to imperfect substitution with their CRESH *substitution/complementarity* parameters, (ρ_i), **Table 4**, (Col.3) - implies *lower* productivity/wages. Over the next decades (*after* 2050) this effect is stabilised (stopped), as *Higher* fertility results in larger *increases* in the *Labor supplies* at *all* ages (i). Moreover, at the *Macro* (aggregate) level we note the simple *proportionality* relation - $\mathbf{W_A}\cdot$ Support ratio - between **Fig. 5** and **Fig. 6**.

The *average annual wage* $\mathbf{W_A}$ for our Labour aggregator (Aggregated Labour supplies), $L_A(t) = L(t)$, is an *exogenous constant* for any *Calendar year* (t) by assumption:

$$W_A(2010) = W_A(2020) = W_A(2030) = W_A(2040) = W_A(2050) = W_A(2070) = W_A(2090) \quad (42)$$

Aggregate wages wL is *any* year allocated to workers by *age* according to (12-15),(25-29). Despite (42) and **Fig. (5,6)**, *annual wages*, $w_i(t)$, of particular *Age groups* (i) or *Generations* are certainly *non-constant* for *Calendar* years, as seen in **Fig. (2a-2e)**. More on this below; cf. **Table 8D**, and Age group wages, $w_i(t)$, as *"shadow values"* (*marginal value-added* : $W_A\,\partial f/\partial L_i$) in (84).

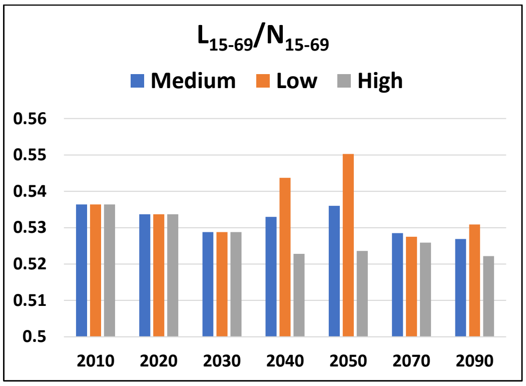
Row 10 provides the *macro* values of (Y/N), (6), in accordance with *Support ratio*, (L/N) (row 7), and *macro* Labour productivity, $Y/L\,(2010) = 581972$ DKK, **Table 3** (row 11). We have not shown (Y/N) graphically over time - being just *proportional* to **Fig. 5** with the *constant*, Y/L (2010). Thus, e.g., **2030**, (Y/N) is 213128 DKK in the *Low* variant, and 205629 DKK in the *Medium* variant, cf. **Table 6** (row 10).

Table 6. Population Age Groups, Labor Supply, LFP and Support Ratios, Incomes per capita					
			Medium Variant	**Low Variant**	**High Variant**
2020	1.	$N_{15\text{-}69}$	3981520	3981520	3981520
	2.	$N_{0\text{-}14}$	941060	897840	984290
	3.	N_{70+}	853060	853060	853060
	4.	N = *Total*	5775640	5732420	5818870
	5.	$L = L_{15\text{-}69}$	2124861	2124861	2124861
	6.	$L/N_{15\text{-}69}$	0.5337	0.5337	0.5337
	7.	L/N	0.3679	0.3707	0.3652
	8.	*wL, Bill.*	929.739	929.739	929.739
	9.	*wL/N,*	160976	162190	159780
	10.	*Y/N*	213884	215512	212314
2030	1.	$N_{15\text{-}69}$	4015690	4015690	4015690
	2.	$N_{0\text{-}14}$	990530	779610	1201460
	3.	N_{70+}	997060	997060	997060
	4.	N = *Total*	6003280	5792360	6214210
	5.	$L = L_{15\text{-}69}$	2123406	2123406	2123406
	6.	$L/N_{15\text{-}69}$	0.5288	0.5288	0.5288
	7.	L/N	0.3537	0.3666	0.3417
	8.	*wL,Bill.*	929.102	929.102	929.102
	9.	*wL/N*	154766	160401	149513
	10.	*Y/N*	205629	213128	198652
2040	1.	$N_{15\text{-}69}$	3970200	3853820	4086590
	2.	$N_{0\text{-}14}$	1033640	752420	1316240
	3.	N_{70+}	1168780	1168780	1168780
	4.	N = *Total*	6172620	5775020	6571610
	5.	$L = L_{15\text{-}69}$	2115947	2095269	2136625
	6.	$L/N_{15\text{-}69}$	0.5330	0.5437	0.5228
	7.	L/N	0.3428	0.3628	0.3251
	8.	*wL,Bill.*	925.839	916.791	934.887
	9.	*wL/N*	149991	158751	142261
	10.	*Y/N*	199292	210919	189002
2050	1.	$N_{15\text{-}69}$	4056090	3751900	4360410
	2.	$N_{0\text{-}14}$	1011340	709050	1337760
	3.	N_{70+}	1231760	1231760	1231760
	4.	N = *Total*	6299190	5692710	6929930
	5.	$L = L_{15\text{-}69}$	2174016	2064811	2283240
	6.	$L/N_{15\text{-}69}$	0.5360	0.5503	0.5236
	7.	L/N	0.3451	0.3627	0.3295
	8.	*wL,Bill.*	951.247	903.464	999.038
	9.	*wL/N*	151011	158705	144163
	10.	*Y/N*	200629	210861	191560
2070	1.	$N_{15\text{-}69}$	4127560	3392170	4916480
	2.	$N_{0\text{-}14}$	1065040	607470	1655720
	3.	N_{70+}	1375060	1375060	1375060
	4.	N = *Total*	6567660	5374700	7947260
	5.	$L = L_{15\text{-}69}$	2181221	1789277	2585545
	6.	$L/N_{15\text{-}69}$	0.5285	0.5275	0.5259
	7.	L/N	0.3321	0.3329	0.3253
	8.	*wL,Bill.*	954.400	782.903	1131.313
	9.	*wL/N*	145318	145665	142353
	10.	*Y/N*	193071	193536	189118
2090	1.	$N_{15\text{-}69}$	4208500	2908130	5752440
	2.	$N_{0\text{-}14}$	1049000	521510	1846430
	3.	N_{70+}	1515320	1475420	1555220
	4.	N = *Total*	6772820	4905060	9154090
	5.	$L = L_{15\text{-}69}$	2217376	1543876	3003654
	6.	$L/N_{15\text{-}69}$	0.5269	0.5309	0.5222
	7.	L/N	0.3274	0.3148	0.3281
	8.	*wL, Bill.*	970.219	675.528	1314.258
	9.	*wL/N*	143252	137721	143571
	10.	*Y/N*	190339	183014	190746

Table 2, 2010: w = 437552 DKK, L= 211472 Labor years, wL= 924.317 Bill. DKK, wL/N = 166615 DKK, L/N= 0.3806
Y/L = 581972 DKK, Y/N = (Y/L)(L/N) = 221499 DKK.

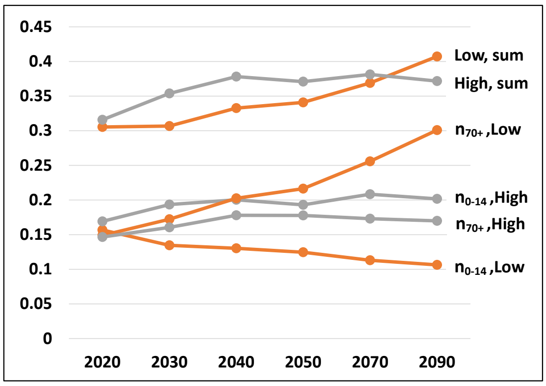
Source: Rows 1-4: United Nations (2015), cf. **Tables (1,2)**; Rows 5-9: **Tables (5a,5b).**

Fig. 3. *Labor Force Participation rates: Medium, Low, High fertility, 2010-2090.*



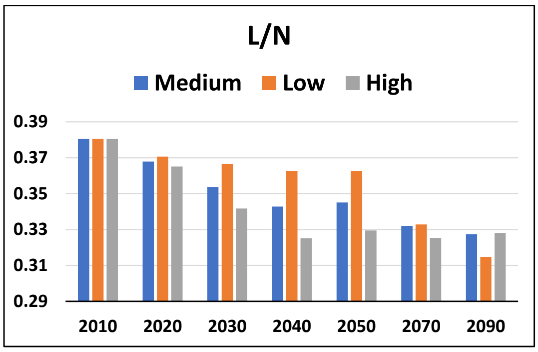
Source: Danish **LFP** $= \frac{\mathbf{L_{15-69}}}{\mathbf{N_{15-69}}} = l_{15-69}(t)$, (2), (4), (39), **Tables (6, 3a, 2)**.

Fig. 4. *Shares, N_{0-14}/N, N_{70+}/N, for Low, High fertility scenario, 2020-2090.*



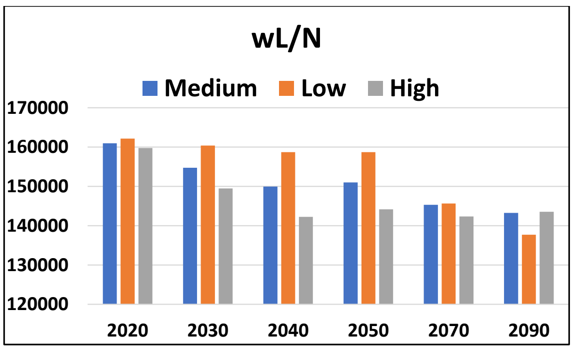
Source: $\mathbf{n_{0-14}}$, $\mathbf{n_{70+}}$, (1), **Tables (1, 6)**.
$sum \equiv n_{0-14} + n_{70+} = 1 - n_{15-69} = \frac{d}{1+d}$

Fig. 5. *Danish Support ratios* - $\mathbf{L/N}$ -



Source: Danish $\frac{\mathbf{L(t)}}{\mathbf{N(t)}}$, (3-6), **Tables (6,3a)**.

Fig. 6. *Wage Income per capita, 2020-2090.*



Source: Danish $W_A(2010) \cdot \frac{\mathbf{L(t)}}{\mathbf{N(t)}}$, **Table 6**.

5 Projected lifetime wage incomes of special cohorts

From the **calendar** (time, t) annual wages, $\mathbf{w_i(t)}$, of labor age groups, $\mathbf{L_i(t)}$, in **Tables (5a,5b)**, we can for a particular **cohort** (T), *extract* the *cohort annual wages*, $\mathbf{w_i^*(T)}$, at each *life-cycle age*, i= 1, 2, , M =11, where, age $i = 1 =$ 15-19, age $i = 2 =$ 20-24, etc.

The **Labor supply** (Labor inputs), $\mathbf{L_i^*(T)}$, of **cohort** (T) at life-cycle *age* (i) is related to the **calendar** Labor supplies, $\mathbf{L_i(t)}$ of *age* group (i), **Tables (5a,5b)**, as follows :

$$L_i^*(T) = L_i(T + [i-1]\,5\,) = L_i(t),\;\; L_1^*(T) = L_1(t)\,;\;\; L_3^*(2020) = L_3(2030) = L_{25-29}(2030)$$

Similarly for $\mathbf{w_i^*(T)}$,[19] *Cohort* (T) annual wage *age* (i) and *Calendar* annual wage $\mathbf{w_i(t)}$:

$$w_i^*(T) = w_i(T{+}[i{-}1]\,5\,) \;\equiv\; w_i(t),\; i = 1, , 11\,;\; w_1^*(T) = w_1(t),\; w_3^*(2020) = w_3(2030) \quad (43)$$

All the results for, $L_i^*(T), w_i^*(T)$, for every **Cohort T** are collected in **Table 7**.

[19]By **Tables (5a,5b)** and (43), a few examples of *extracting*, $w_i^*(T)$, $L_i^*(T)$, **Table 7**, are :

Table 7. **Medium** Variant : Example - Cohort wages, $w_i^*(T)$, of Cohort, $\mathbf{T = 2010}$,

Table 5a: $w_{15-19}^*(2010) = w_{15-19}(2010) = 187005\,; w_{25-29}^*(2010) = w_{25-29}(2020) = 333172$

Table 5b: $w_{45-49}^*(2010) = w_{45-49}(2040) = 467867;\; w_{55-59}^*(2010) = w_{55-59}(2050) = 444392$ **Table 7**. **Medium** Variant : Example - Cohort wages, $w_i^*(T)$, of Cohort, $\mathbf{T = 2015}$,

Table 5a: $w_{15-19}^*(2015) = w_{15-19}(2015) = 187053\,; w_{30-34}^*(2015) = w_{30-34}(2030) = 393930$

Table 5b: $w_{40-44}^*(2015) = w_{40-44}(2040) = 475811;\; w_{50-54}^*(2015) = w_{50-54}(2050) = 459807$

Table 7. **Medium** Variant : Example - Cohort wages, $w_i^*(T)$, of Cohort, $\mathbf{T = 2020}$,

Table 5a: $w_{15-19}^*(2020) = w_{15-19}(2020) = 188933\,; w_{25-29}^*(2020) = w_{25-29}(2030) = 340041$

Table 5b: $w_{35-39}^*(2020) = w_{35-39}(2040) = 454562;\; w_{65-69}^*(2020) = w_{65-69}(2070) = 468895$

Table 7. **High** Variant : Example - Cohort wages, $w_i^*(T)$, of Cohort, $\mathbf{T = 2030}$,

Table 5a: $w_{15-19}^*(2030) = w_{15-19}(2030) = 191356\,;$

Table 5b: $w_{35-39}^*(2030) = w_{35-39}(2050) = 484008;\; w_{55-59}^*(2030) = w_{55-59}(2070) = 493656$

Table 7. **Low** Variant : Example - Cohort wages, $w_i^*(T)$, of **Cohort, $\mathbf{T = 2035}$**,

Table 5b: $w_{20-24}^*(2035) = w_{20-24}(2040) = 279471;\; w_{50-54}^*(2035) = w_{50-54}(2070) = 459807$

Thus **Medium** variant size of $w_i^*(2010)$ for the **Cohort 2010** at age 25-29 is 333172 (**Table 7**, row 6 column 2). This number was seen as $w_i(2020)$ for 25-29 year old in 2020 (**Table 5a**, column 5, row 6).

For the **Cohort 2020**, the sizes of $w_{15-19}^*(2020)$, $w_{25-29}^*(2020)$, $w_{35-39}^*(2020)$, $w_{45-49}^*(2020)$ in **Table 7** are the sizes seen in **Tables 5a, 5b**, (column 5) for, $w_{15-19}(2020)$, $w_{25-29}(2030)$, $w_{35-39}(2040)$, $w_{45-49}(2050)$, respectively.

The *cohort* annual wages, $w_i^*(T)$, and *cohort* labor supplies, $L_i^*(T)$, in **Table 7** can be *summed* to generate the *Total Life* Wage Income of **Cohort** $(T) : w^*(T)L^*(T)$, where Labor Supply, $\mathbf{L^*(T)}$, is the *Total Life Time* Labor Supply of **Cohort** (T), and $\mathbf{w^*(T)}$ is the *Average (Life)* Annual Wage of **Cohort** (T),. i.e., as defined in accordance with:

$$w^*(T)L^*(T) = \sum_{i=1}^{M} w_i^*(T)L_i^*(T)\,;\; L^*(T) = \sum_{i=1}^{M} L_i^*(T)\,;\; w^*(T) = \frac{\sum_{i=1}^{M} w_i^*(T)L_i^*(T)}{L^*(T)} \qquad (44)$$

These *Longitudinal* (Cohort) Labor *supplies*, $L_i^*(T)$, [life-cycle ages (i)], *Longitudinal* annual *wages*, $w_i^*(T)$, and *Life Time* Cohort Labor supply, $L^*(T)$, making the **Average** *Annual Wage*, $\mathbf{w^*(T)}$,[20], (44), are shown in **three** demographic *Variants* for **six** Labor **Cohorts**, **T**, (45), in **Table 7** - where, $\mathbf{T - 15 = t}$, is *Birth year* (**t**) of the *youngest*

$$T = \text{ 2010, 2015, 2020, 2025, 2030, 2035}\,;\; t = \text{1995, 2000, 2005, 2010, 2015, 2020} \qquad (45)$$

Generation (**t**), which enter the Labor **Cohort** (**T**). Thus Cohort **T=2035** (born 2020) (45) starts working t= 2035 and is retired in year, t = 2090 - the *end year* of **Table 5b**.

We gave above a few examples on how to extract (translate) information from Tables (5a,5b) to Cohorts in Table 7. Similarly many other numbers, $w_i^*(T)$, $L_i^*(T)$, in Table 7 can be traced back to Tables (5a,5b). However, for *every Cohort* (T), (45) - all longitudinally Cohort variables of $w_i^*(T)$, $L_i^*(T)$, exhibited in **Table 7**, contain *much more* information (numbers) than available in **Tables (5a,5b)**, as evidently many *needed intermittent* calendar *years* (2015, 2025, 2035, 2045, etc.) are *not* shown in **Tables (5a,5b)**. Hence the entire **Table 7** has been obtained by *completing* all the additional *calculations* needed (but not shown) to **extend** the **Tables (5a,5b)**.

It is important to fully realize, however, that it is **all** 11 *overlapping age-group* (cross-section) calendar wages, $\mathbf{w_i(t)}$, of *many* **Calendar** years (**t**) that **generate** - by *equation*, (43) - the relevant **longitudinal** annual wages, $\mathbf{w_i^*(T)}$, i= 1, ,11, of Labor **Cohort T** (and their **Generation**) through a *working life-cycle* of **55 years** in 11 *age-groups (i).*

[20] The **Average** lifetime **Wage** (Earnings) of **Cohort T**, $\mathbf{w^*(T)}$, (44), differs from Lifetime Earnings of a *Cohort worker*, optimally *accumulating* human capital (education, experience) and *rentals (wages)* during fixed lengths of working-life (*ages/years*). On shape of the life-cycle (*Age-wage profiles*) of such Cohort, see Rosen (1972, p.330; 1976, p.52), Welch (1979. p.79), and Berger (1984, p.590; 1985, p.572).

Table 7. Age-group Wages and Lifetime Wage Incomes of Cohorts, entering Labor market at age 15, size $N_{15\text{-}19}$, in year : 2010, - - -, 2035 ; Three Variants : Denmark

Medium	Cohort 2010 ($N_{15\text{-}19}$, 353109)			Cohort 2015 ($N_{15\text{-}19}$, 358260)			Cohort 2020 ($N_{15\text{-}19}$, 338740)			Cohort 2025 ($N_{15\text{-}19}$, 345300)			Cohort 2030 ($N_{15\text{-}19}$, 309940)			Cohort 2035 ($N_{15\text{-}19}$, 322890)		
	w_i^*	L_i^*	$w_i^*L_i^*$	w_i^*	L_i^*	$w_i^*L_i^*$	w_i^*	L_i^*	$w_i^*L_i^*$	w_i^*	L_i^*	$w_i^*L_i^*$	w_i^*	L_i^*	$w_i^*L_i^*$	w_i^*	L_i^*	$w_i^*L_i^*$
15-19	187005	30199	5.647	187053	30640	5.731	188933	28970	5.473	187560	29531	5.539	191356	26507	5.072	190378	27615	5.257
20-24	265673	128659	34.181	267205	124262	33.203	269161	117761	31.697	267744	119962	32.119	274181	108172	29.659	272882	112502	30.700
25-29	333172	214793	71.563	335522	207736	69.700	340041	197287	67.086	339063	200851	68.101	350480	181862	63.739	347869	188860	65.699
30-34	390937	286458	111.987	393930	277315	109.243	400898	263751	105.737	400119	268424	107.402	413462	243728	100.772	409479	252864	103.542
35-39	437674	308193	134.888	444123	298552	132.594	454562	284214	129.193	453724	289218	131.225	472578	263051	124.312	465508	272550	126.874
40-44	466698	311857	145.543	475811	302296	143.836	490602	288018	141.302	488388	293087	143.140	511680	266774	136.503	502425	276170	138.755
45-49	467867	310995	145.504	478120	301650	144.225	491733	287586	141.416	487251	292640	142.589	510103	266502	135.943	501927	275833	138.448
50-54	453094	303977	137.730	459807	295048	135.665	468150	281456	131.764	464441	286523	133.073	482564	261075	125.985	476753	270298	128.865
55-59	444392	283113	125.813	447655	275079	123.140	453593	262658	119.140	451575	267571	120.828	465032	244016	113.475	461405	252770	116.629
60-64	471668	139335	65.720	474002	135561	64.256	478816	129609	62.059	477769	132136	63.130	487602	120667	58.837	484390	125061	60.578
65-69	461214	28526	13.157	464096	27807	12.905	468895	26634	12.489	467990	27192	12.726	476666	24874	11.857	472797	25811	12.203
Total		2.346105	991.734		2.275946	974.499		2.167944	947.354		2.207135	959.873		2.007228	906.155		2.080334	927.551
Lifetime	w^* = 422715			w^* = 428173			w^* = 436983			w^* = 434895			w^* = 451446			w^* = 445866		
Low	Cohort 2010 ($N_{15\text{-}19}$, 353109)			Cohort 2015 ($N_{15\text{-}19}$, 358260)			Cohort 2020 ($N_{15\text{-}19}$, 338740)			Cohort 2025 ($N_{15\text{-}19}$, 345300)			Cohort 2030 ($N_{15\text{-}19}$, 309940)			Cohort 2035 ($N_{15\text{-}19}$, 279720)		
15-19	187005	30199	5.647	187053	30640	5.731	188933	28970	5.473	187560	29531	5.539	191356	26507	5.072	195761	23923	4.683
20-24	265673	128659	34.181	267205	124262	33.203	269161	117761	31.697	267744	119962	32.119	273955	108172	29.634	279471	98088	27.413
25-29	333172	214793	71.563	335522	207736	69.700	340041	197287	67.086	338724	200851	68.033	348891	181862	63.450	357914	165624	59.279
30-34	390937	286458	111.987	393930	277315	109.243	400498	263751	105.632	398306	268424	106.915	408972	243728	99.678	417453	222601	92.925
35-39	437674	308193	134.888	443602	298552	132.438	452058	284214	128.481	447615	289218	129.458	461321	263051	121.351	471752	240440	113.428
40-44	466068	311857	145.347	472725	302296	142.903	482722	288018	139.033	474304	293087	139.012	489117	266774	130.484	501493	243993	122.361
45-49	464833	310995	144.561	470441	301650	141.909	477552	287586	137.337	465765	292640	136.301	478578	266502	127.542	490299	243856	119.562
50-54	446993	303977	135.876	448854	295048	132.433	451227	281456	127.001	440909	286523	126.331	450183	261075	117.532	458453	239010	109.575
55-59	436048	283113	123.451	435060	275079	119.676	435721	262658	114.446	427892	267571	114.491	434226	244016	105.958	440843	223576	98.562
60-64	462208	139335	64.402	460730	135561	62.457	460833	129609	59.728	455043	132136	60.128	460166	120667	55.527	465324	110690	51.507
65-69	448299	28526	12.788	446666	27807	12.420	446591	26634	11.895	441658	27192	12.010	446861	24874	11.115	451423	22853	10.316
Total		2.346105	984.690		2.275946	962.114		2.167944	927.807		2.207135	930.337		2.007228	867.343		1.834654	809.612
Lifetime	w^* = 419713			w^* = 422731			w^* = 428173			w^* = 421513			w^* = 432110			w^* = 441289		
High	Cohort 2010 ($N_{15\text{-}19}$, 353109)			Cohort 2015 ($N_{15\text{-}19}$, 358260)			Cohort 2020 ($N_{15\text{-}19}$, 338740)			Cohort 2025 ($N_{15\text{-}19}$, 345300)			Cohort 2030 ($N_{15\text{-}19}$, 309940)			Cohort 2035 ($N_{15\text{-}19}$, 366070)		
15-19	187005	30199	5.647	187053	30640	5.731	188933	28970	5.473	187560	29531	5.539	191356	26507	5.072	185815	31308	5.817
20-24	265673	128659	34.181	267205	124262	33.203	269161	117761	31.697	267744	119962	32.119	274411	108172	29.684	267355	126915	33.931
25-29	333172	214793	71.563	335522	207736	69.700	340041	197287	67.086	339407	200851	68.170	352103	181862	64.034	339705	212103	72.052
30-34	390937	286458	111.987	393930	277315	109.243	401305	263751	105.845	401972	268424	107.899	418065	243728	101.894	403417	283127	114.218
35-39	437674	308193	134.888	444651	298552	132.751	457112	284214	129.918	459955	289218	133.027	484008	263051	127.319	461345	304660	140.553
40-44	467334	311857	145.741	478945	302296	144.783	498618	288018	143.611	502659	293087	147.323	534278	266774	142.531	504420	308347	155.536
45-49	470949	310995	146.463	485933	301650	146.582	506101	287586	145.548	508771	292640	148.887	541142	266502	144.215	512678	307803	157.804
50-54	459316	303977	139.621	470928	295048	138.946	485100	281456	136.534	487510	286523	139.683	513620	261075	134.093	492711	301586	148.595
55-59	452915	283113	128.226	460327	275079	126.626	471106	262658	123.740	474117	267571	126.860	493656	244016	120.460	478870	281964	135.024
60-64	481322	139335	67.065	487115	135561	66.034	495901	129609	64.273	498625	132136	65.886	512222	120667	61.808	500568	139428	69.793
65-69	473973	28526	13.521	480655	27807	13.366	489364	26634	13.034	491620	27192	13.368	503417	24874	12.522	491326	28768	14.134
Total		2.346105	998.904		2.275946	986.966		2.167944	966.757		2.207135	988.761		2.007228	943.633		2.326009	1047.460
Lifetime	w^* = 425771			w^* = 433651			w^* = 445933			w^* = 447984			w^* = 470117			w^* = 450325		

Source : United Nations (2015), New York 2015 ; *Cohorts* ($N_{15\text{-}19}$) in years, 2010, 2015, 2020, 2025, 2030, 2035. *Source* : **Tables (5a,5b).**

w_i^* : Cohort wages, *age (i),* DKK ; L_i^* : Cohort Labor supply ; $w_i^*L_i^*$: Cohort Wage Income, Billion DKK ; w^*L^*= Cohort Life Wage Income ; w^* : Average Cohort Life Annual Wage, DKK. Comparison, e.g., with Table 2, 2010 : L (2010) = 2.112472 = Total Labor Supply (Total Labor years), Million ; Total Wage sum, wL = 924.317 Billion DKK ; w (2010) = 437552 DKK.

For $\mathbf{T = 2010}$: L^*_{15-19} are born in the *calendar years* (t): 1991-1995 (*Generation*). Then : L^*_{20-24} refer to 2015-2020; L^*_{25-29} refer to 2021-2025; L^*_{30-34} refer to 2026-2030 ; L^*_{35-39} refer to 2031-2035; L^*_{40-44} refer to 2036-2040; L^*_{45-49} refer to 2041-2045; L^*_{50-54} refer to 2046-2050; L^*_{55-59} refer to 2051-2055; L^*_{60-64} refer to 2056-2060; L^*_{65-69} refer to 2061-2065.

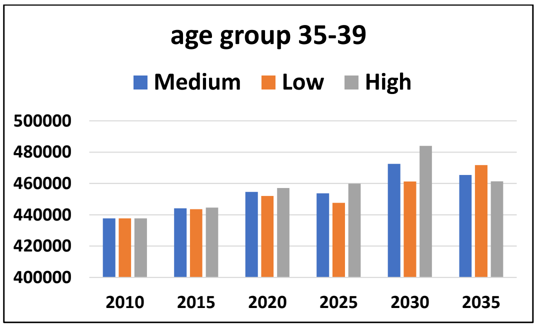
For $\mathbf{T = 2015}$: L^*_{15-19} born *calendar years* (t): 1996-2000 (*Generation Z*). Then L^*_{20-24} refer to 2020-2025 ; L^*_{65-69} refer to 2066-2070. For $\mathbf{T = 2020}$: L^*_{15-19} born *calendar years* (t): 2001-2005. Then L^*_{20-24} refer to 2026-2030 ; L^*_{65-69} refer to 2071-2075. For $\mathbf{T = 2030}$: L^*_{15-19} born *years* (t): 2011-2015. Then L^*_{20-24} refer to 2036-2040 ; L^*_{65-69} refer to 2081-2085.

Fig. 7.

Annual wages of age group 35-39,

$w^*_{35-39}(T)$, *for 6 Cohorts 2010-2035*

Medium, Low, High fertility



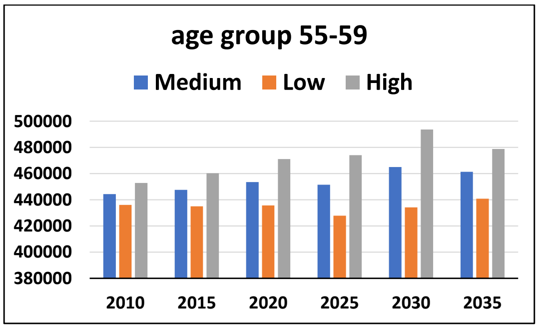
Source: $\mathbf{w}^*_{\mathbf{i}}(\mathbf{T})$, (43), $i = 35 - 39$, T, (45) , in **Table 7**.

Fig. 8.

Annual wages of age group 55-59,

$w^*_{55-59}(T)$, *for 6 Cohorts 2010-2035*

Medium, Low, High fertility

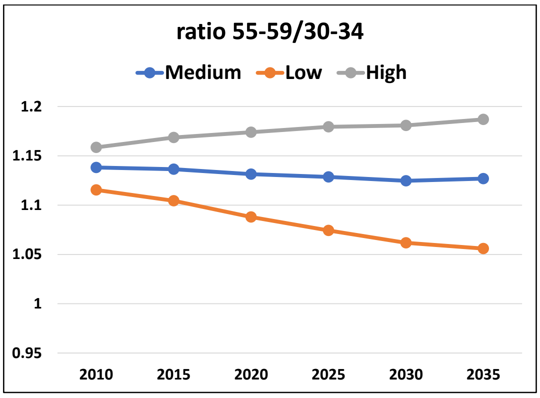


Source: $\mathbf{w}^*_{\mathbf{i}}(\mathbf{T})$, (43), $i = 55 - 59$, T, (45) , in **Table 7**.

Fig. 9.

Ratio of the Annual Wages between stages (old/young) for 6 Cohorts T, 2010-2035

Medium, Low, High fertility.

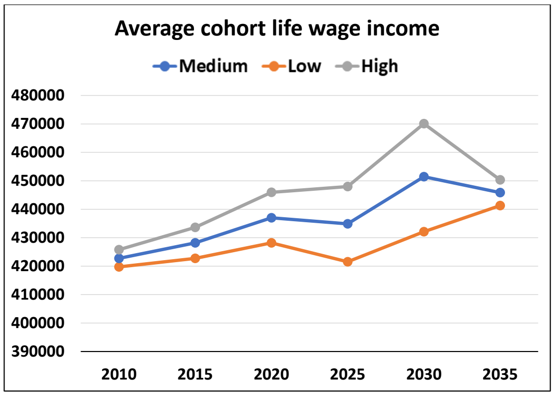


Source: $w^*_{55-59}(T)/w^*_{30-34}(T)$ in **Table 7**, as obtained from : $\mathbf{w}^*_{\mathbf{i}}(\mathbf{T})$ in **Fig. 7 - 8**.

Fig. 10.

Average (annual wage) Life - all ages (i) - Income, $w^*(T)$, *for 6 Cohorts T, 2010-2035*

Medium, Low, High fertility.



Source: $\mathbf{w}^*(\mathbf{T})$, (44), (45), in **Table 7.**

The *sizes* of the *Generations/Cohorts* are seen in **Table 7** (second row) as, $\mathbf{N_{15-19}(T)}$, **Table (5a,5b)**, i.e., specific *size* of *Population* age *group* (15-19) - *youngest* (15) entering Labor market, $\mathbf{L^*_{15-19}(T)}$ - in the *years*, (45) : T = 2010, 2015, 2020, 2025, 2030, 2035.

In the **Medium** variant for example, $\mathbf{N_{15-19}(2020)}$ = 338740 *people* aged 15-19 in **2020**, which is also seen in **Table 5a** (row 1, column 2). For the same variant, $\mathbf{N_{15-19}(2030)}$ = 309940 can be seen in **Table 5a**, (row 13, column 2). The $\mathbf{N_{15-19}(2035)}$ = 322890 (not shown in **Tables 5,6**) will be 80-85 years of age in 2100, which is the last year of the United Nations population projections. The corresponding rows with $\mathbf{N_{15-19}(T)}$ for the *Low* and *High* variants are given further down in **Table 7**.

Generation *sizes* $\mathbf{N_{15-19}(T)}$ - and Cohort *Lifetime* Labor *supply*, $\mathbf{L^*(T)}$, (44) - are *equivalent* for **all** variants in all years, *except* for the last, *Cohort* (2035), since the *fertility change* commences in 2020 and takes until 2035 to be reflected in Cohort Labor, $\mathbf{L^*(T)}$.

Figures (7, 8) show **Cohort (T)** *annual wages* for workers in the second half of the 30s, $\mathbf{w^*_{35-39}(T)}$, and for workers in the second half of the 50s, $\mathbf{w^*_{55-59}(T)}$. For the youngest age group, **Fig. 7**, significant changes are found when comparing the 2030 and the 2035 **Cohorts**. Here we find a strong impact in the *High* fertility case, where relative increase in younger workers, $\mathbf{L^*_{35-39}(T)}$, has a *depressing* effect on $\mathbf{w^*_{35-39}(T)}$. The counterpart to this is shown clearly in **Fig. 8**, where *High* fertility *improves* the position of *older* workers, $\mathbf{w^*_{55-59}(T)}$, for *all* 6 **Cohorts**, more so for the *2030* and *2035* **Cohorts**.

Fig. 9 presents an alternative illustration of how the *ratio*, old/young *annual wages* are affected for 6 **Cohorts** in *three* Fertility scenarios. Not surprisingly, *old* workers, $\mathbf{w^*_{55-59}(T)}$, are much *better* off relatively in the *High* compared to the *Low* fertility case.

Fig. 10 shows **Cohort** Average (*Life-time*) annual wage, $\mathbf{w^*(T)}$, for **Cohorts T**, 2010 to 2035, covering their *full* working life, summing up their wages at different ages (life cycle) in **Fig. 7-8**. Thus **Cohort** *2035* consists of workers 15-19 years $\mathbf{L^*_{15-19}}(2035)$ in year 2035, and of workers $\mathbf{L^*_{65-69}}(2035)$ retiring during 2086-2090, and living as 70-74 years old, $\mathbf{N_{70+}}(2015)$, in 2090. Even though the **Cohort** *2035* had *lowest* $\mathbf{w^*_{35-39}}(2035)$ with High fertility, then much better wages later as e.g., $\mathbf{w^*_{55-59}}(2035)$, ensured that the *Average* (Life time) wage, $\mathbf{w^*}(2035)$, were *highest* with High fertility. The importance for any Cohort $\mathbf{w}^*(T)$ of having *many* and *large* surrounding (*cooperating*) cohorts as *co-workers* for the particular *Cohort T* (Generation) during its full working life (period). The *explicit* wage formula of $\mathbf{w}^*(\mathbf{T})$, (44), with the *analytic* CRESH forms (40-41) of *wage complementarity* emphasize such interaction (mutual interdependence) behind $\mathbf{w}^*(\mathbf{T})$.

Fig. 10 compared future prospects of **Cohorts** using the *Average* Lifetime annual wages of the Cohort, $\mathbf{w}^*(\mathbf{T})$, shown in **Table 7** (Row - Lifetime - in each variant). The value of $\mathbf{w}^*(\mathbf{T})$ is *highest* for the *smallest Cohort*, (**2030**), in *Medium, High* variants. For the *Medium* variant: $\mathbf{w}^*$(2010)=422715, $\mathbf{w}^*$(2015)=428173, $\mathbf{w}^*$(2020) =436983, $\mathbf{w}^*$(2025) =434895, $\mathbf{w}^*$(2030)=451446, $\mathbf{w}^*$(2035)=445866 - depicted in **Fig. 10** (*blue*). Thus Medium it Cohort (2030) has $\mathbf{w}^*$(2030) as 5.2 *percent* higher[21] than $\mathbf{w}^*$(2015). For *Low* Fertility variant, the *highest* $\mathbf{w}^*$ is also seen for the *smallest Cohort* (T=1935).

The relationships between **Cohort size** - measured by Cohort Labor supply, $\mathbf{L}^*(\mathbf{T})$ - and *Average* (life-time) annual *wage*, $\mathbf{w}^*(\mathbf{T})$, are illustrated for the three fertility scenarios in **Fig. 11**. The *slopes* are *negative* with Cohort Average (life-time) wages *increasing* with *decreasing Cohort size.* The *exceptions* are found for the **High** fertility cases where Average (life-time) annual wage is significantly *higher* for *large* 2025 and 2035 *Cohorts*, reflecting the positive wage impacts increased Labor supply of *co-workers* in other cohorts.

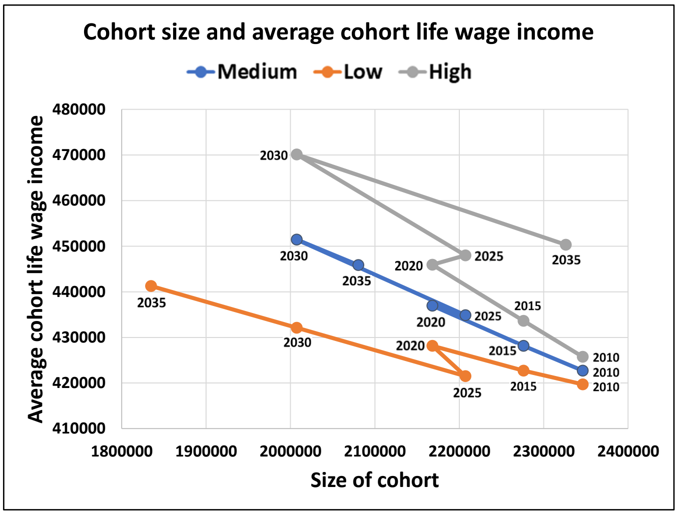
Indeed **Cohort** differences in $\mathbf{w}^*(\mathbf{T})$, **Fig. 10-11**, for all three demographic variants, are - with *same parametric* CRESH model for $\mathbf{w}_\mathbf{i}^*(\mathbf{T})$, **Fig. 7 - 9** - fully explained by wage *complementarity differences* that Cohort Labor supplies, $\mathbf{L}^*(\mathbf{T})$,[22], (44) are exposed to.

In section 2.2, we saw in **Table 2** (col.2) some positive/negative Population **echo's** of the *sizes* of earlier generations, and in section 4.1, we saw for *calendar years* in **Table 5a**, **Fig. (2a-2c)**, the *annual wage*, $\mathbf{w_i(t)}$, effects of belonging to the *small* (early) *Millennial* generations, (1981-1985), (1986-1990). We have not shown (calculated) the *Life time* wage income, $\mathbf{w}^*(\mathbf{T})$, of first two *Millennial* generations (*Cohorts*, T=2000, T=2005), but they should be *high* as $\mathbf{w}^*(\mathbf{2000})$, $\mathbf{w}^*(\mathbf{2005})$ - not shown in **Fig. 10-11**. But $\mathbf{w}^*(\mathbf{T})$ of the last *Millennial* generation and the first *Z* - generation (*Cohorts*, T=2010, T=2015) are as $\mathbf{w}^*(\mathbf{2010})$, $\mathbf{w}^*(\mathbf{2015})$ in **Fig. 10-11** - being *lower* than $\mathbf{w}^*(\mathbf{T})$ of more *future* Cohorts.

[21] The $\mathbf{N_{15-19}(2030)}$, is is 13.5 *percent* smaller than $\mathbf{N_{15-19}(2015)}$, cf. **Table 7**.

[22] The size of these $\mathbf{w}^*(\mathbf{T})$ effects depends on the degree of labor substitutability reflected in the parameter ρ_i. If all $\rho_i = 1$, there is perfect labor substitution and *sizes* of Cohort/Labor supplies have no effect on its *own* relative (absolute) wages nor affect the relative (absolute) wage of *other* Cohorts.

Fig. 11. Life Time **Cohort Labor** Supply, **L***(**T**), ("Cohort size"), and **Cohort Average** Life Income (*Annual* wage), **w***(**T**) - for the *six Cohorts*, 2010-2035, in *three* variants : Medium, Low, High **fertility**.

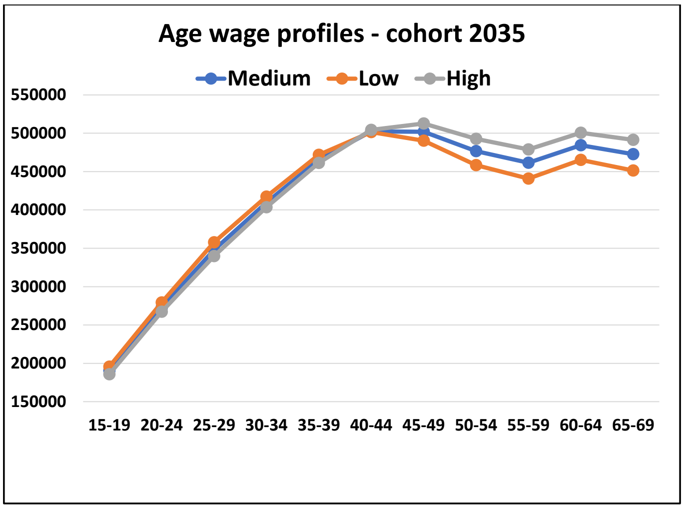


Source: Six numbers of **L***(**T**), and, **w***(**T**), (44), (45), seen (bottom) in **Table 7.**

Fig. 12. Annual wages - **Age wage profile** - for **Cohort** 2035,

$\mathbf{w}_\mathbf{i}^*$(**T**), i = 1,2, ,11, Cohort T= 2035, **Generation**, t= 2020,

- Medium, Low, High **fertility**.



Source: $\mathbf{w}_\mathbf{i}^*$(**T**), (43), i=1, 2, , 11 $\equiv$ i= 15-19, 20-24, ,65-69,

T = 2035, t= 2020, (45), from **Table 7**, (last **Cohort**, RHS).

We may trace some **echo** of small *first Millennial* Generation (1981-1985), *Cohort* (T=2005), on *Life time* wage income ($\mathbf{w}^*$) of their *descendants* (progeny). Generation (1981-1985) is not exclusively - but it is the main *Progenitor* of Generation (2011-2015), Cohort T = 2030, and we do see an **echo** of first *Millennial* Cohort (T=2005) in *Progenitor* Cohort (T=2030) - as reflected in $\mathbf{w}^*(\mathbf{2030})$ - which indeed is the *highest* ($\mathbf{w}^*$) in **Table 7**, **Fig. 10-11**, with *smallest* sizes of $\mathbf{N_{15-19}}(\mathbf{2030}) = 309940$, or $\mathbf{L}^*(\mathbf{2030}) = 2.007.228$.

Fig. 12 shows the **longitudinal** *annual wages*, $\mathbf{w_i^*}(\mathbf{T})$, to *all ages* (i) (*life cycle*) of the **Cohort**, T = 2035, for three fertility scenarios. Annual wages peak at *ages* 40-44, independently of the fertility scenarios. *After* this age, 40-44, *differences* in **fertility** has a clear impact with *higher* annual wages for *older* workers in the *high* fertility case, reflecting the scarcity of the older workers together with ample supplies of younger workers. The *shape* of the **Age profile** of annual wages, $\mathbf{w_i^*}(\mathbf{T})$, **Fig. 12**, applies *qualitatively* to *any* **Cohort** T in **Table 7** (all *vertical* wage columns of $\mathbf{w_i^*}(\mathbf{T})$, to the *left* of Cohort 2035).

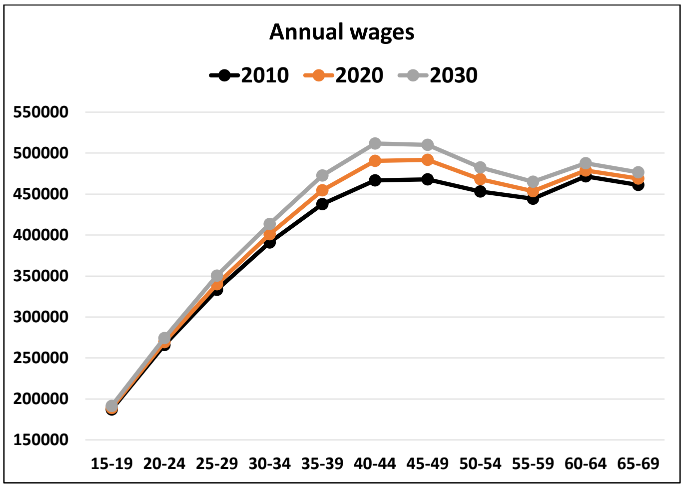
In **Fig. 13**, we show the *Age-wage profiles*, $\mathbf{w_i^*}(\mathbf{T})$, $i = 1, 2, , M = 11$ for the *three* Cohorts, $T = 2010, 2020, 2030$, *Medium* fertility - already seen with their *Life time*, $\mathbf{w}^*(\mathbf{T})$, for $T = 2010, 2020, 2030$ (on *blue* line) in **Fig. 10**. Hence **Fig. 13** demonstrate that e.g., that the largest *Life time*, $\mathbf{w}^*(\mathbf{2030})$, in **Fig. 10** also have the largest $\mathbf{w_i^*}(\mathbf{T})$ at *any* stage (all *ages*), (i), during entire working life (15-69).

We saw in **Fig. (2d)** that the smallest Generation (1981-1985) had - as the age group (45-49) in *calendar* year 2030 - the highest wages (above normal). Such above normal wages in calendar year 2010 is not just a temporary effect - but become a permanent effect - of being a small generation as (1981-1985). Thus such *permanent* wage effect of the small[23] *Generation* (2011-15), **Cohort**, T = 2030, is seen for *all ages* (i) in **Fig. 13**.

[23]The age composition of the labor force varies much over time due to demographic changes. The large post-war *Generations*, born 1946-1964 (defining American "baby boomers"), included *four* 5-year *age groups*, from the leading edge group (1946-50), *peak* in (1956-60), to trailing edge group (1961+), cf. Freeman (1979, p.289, Easterlin et al. (1990, p.281). Danish "baby boomers" refer to *decade* (1941-50).

The economic effects of several large "baby boom generations" (age-groups) are explored extensively in economic/demographic literature. "*Twist* [shift/rotation] in male *age-wage profiles* in late 1960s and early 1970s" (relative *low* earnings of younger workers) have empirically been attributed, Freeman (1979, p.315), Easterlin (1978, p.401), as impacts of *large* "baby boom" generations (1946-60) - the opposite of *small* generations (Y, 1981-90) effects, *calendar* year "*twists*", **Fig. 2d** - or as *life-time* impacts, **Fig. 13.**

Figure 13. Age-wage profiles, $\mathbf{w_i^*(T)}$, i= 1, 2, , M, age group : i = 1 = 15-19, i = 11 = 65-69 - for the **Cohorts**, T = 2010, 2020, 2030 - **Medium** fertility variant.



Source: Cohort wages, $\mathbf{w_i^*(T)}$, (43), T = 2010, 2020, 2030, **Table 7**, *horizontal* top.

The overall **Age-wage** profiles as **Fig. (12,13)**, **Fig. (2d,2e)**, hold *generally* for Cohorts and Calendar years, and the **shape** of such *Age-wage profiles* are determined by the CRESH parametric *Labor Aggregator*, (7-8), (12-14), (15), or obtained by the dual CRESH *Age-Wage* [**Inverse** Labor demand] $\mathbf{w_i}$ - *form*, (25-29), (81), (84), (92-93).

6 Population, Division of Labor, and Wages

Demography, Population, Labor Allocation, Wages, and National Income per capita are subjects of classic fields and studies in Political Economy/Economics. Let us end with a few literature comments provided for both *inductive* and *deductive* aspects of this paper. For this purpose, it is useful to recall the macro relations and the ratios in (5-6), (25),

$$\frac{Y}{N} = \frac{Y}{L}\cdot\frac{L}{N} \;;\; \frac{L}{N} = \sum_{i=1}^{11} l_i \cdot n_i \;=\; l_{15-69}\cdot n_{15-69}\;;\; \frac{W}{N} = \frac{W}{L}\cdot\frac{L}{N} = W_A\cdot\frac{L}{N} \;;\; W_A\, L_A = \sum_{i=1}^{M} w_i L_i \equiv W \quad (46)$$

As to *proportions* in the Per Capita National Income ("Wealth of Nation") *identity*, (46), Smith (1790,1961, p.1) opens with the statement: "The annual labour of every nation is the fund, which supplies it with all the necessaries and conveniences of life. - According therefore, this *produce* [product, output, Y] bears a greater or smaller *proportion* [Y/N] to number [N] of those who are to consume it. But this *proportion* [Y/N] must in every nation be regulated by *two* different circumstances : 1. the *skill, dexterity, and judgement* with which its labour is generally applied [*Labor productivity*, Y/L] 2. the *proportion* [L/N] between the number of those who are *employed* in *useful labor* [L] and those not so employed [N-L]. - The abundance or scantiness of this *supply* [per-capita produce, Y/N] seems to depend more upon the *former* [Y/L] of those two circumstances than upon the *latter*" [L/N] (italics ours).

The *Employment/Population* ("**Support**") ratio, (**L/N**), (bounded above by one[24]) is always much lesser than one as the numerical *size* of (L/N), (46), is by definition the product of **LFP** $= l_{15-69}$ (interval: 0.5-0.6), and the Working population share, $\mathbf{n_{15-69}}$ (interval: 0.7-0.6), i.e., e.g., $L/N = l_{15-69} \cdot n_{15-69} = 0.38$, cf. (5-6), and **Tables (1,2,6)**.

As to (Y/L), Smith (1790, p.7) says: "The greatest improvements in the *productive powers* of *labour*, and the greater part of the *skill, dexterity, and judgement* with which is anywhere directed, or applied, seems to have been the *effects* of the *division of labour*. - p.11: This great increase of the *quantity* [Output] of *work* [Labor productivity,**Y/L**] which, in consequence of the *division of labour*, the *same* number of people [L] are capable of performing, is owing to *three* different circumstances 1. the increase of *dexterity* in *every* particular workman 2. the *saving* of the *time* which is commonly lost in passing from one species of work to another 3. the invention of a *great number* of *machines* $[K, K_J]$ which facilitate and abridge labour, and enable *one man* to do the work of many"(italics ours).

The **Productive** *powers* of *Labor* by the **Division of Labor** (the notions used by Smith above), is economically and conceptually expressed with : **Production functions**, $Y = F(L, K) = L \cdot f(K/L) \equiv L \cdot f(k)$, or as, $F\left(Y, L_I, L_{II}, K_{III}, K_{IV}, K_V\right) = 0$, cf. (34).

Production (Division of Labor) by *different qualities* of *workers* according to skills [education, training], dexterity and judgement [*age/maturity/experience*] provide the framework for analyzing differences in *factor prices* - earnings *structures*, Smith (1790, p.111)[25]: "Pecuniary *wages* and *profits* [rentals of machinery], indeed, are everywhere in Europe extremely different according the *different employments* of *labour* and *stock*"(italics ours).

As to Pecuniary *wage distributions* for the *complete* set of *demographic age-groups* of the *National Labor* supply - making *Total Wages* (W), (46), in the *National* income (Y),

[24]Modern Growth Theory and Macroeconomics, cf. standard exposition, Solow (2000), Romer (2019), do not allow in any of the models for the distinctions between Labor (L) and Population (N).

[25]As to the wage structure analysis in Smith (1790, Ch. 10), see Katz and Autor (1999, p.1464).

Table 3, our CRESH Labor **Aggregator**, $L_A = f(L_1, L_2, \ldots, L_M)$ - and **wage generator** by its *derivatives*, (27-29) - *formed* the annual *wage distributions* : **Calendar** year *Age-wage profiles* in **Tables 5a-5b** - and implied the **Cohort** *Life time* wages in **Table 7**.

It must be emphasized that the calculated results in **Tables (5a-5b, 7)** are **not** dealing with a *pure* Labor Economy, using only distinct Labor inputs. As stressed by Smith above, *Output* (Y) by 'Division of Labor' and *Labor productivity*, $\frac{Y}{L} = y$, involved '*machinery*' [K] in *Production* functions, $Y = F(L, K)$, $y = f(k)$; let, $p \cdot \frac{\partial Y}{\partial L}(k) = W_A\,(k)$:

$$w = p \cdot \frac{\partial Y}{\partial L} \; ; \; r = p \cdot \frac{\partial Y}{\partial K} \; ; \; w = p \cdot \frac{\partial Y}{\partial L} = p \cdot y - r \cdot k \;\equiv\; W_A \, ; \;\; k = 2.72 \; : \; W_A\,(2.72) \; = \; 582 - 0.053 \cdot 2.72 = 438 \quad (47)$$

where actual numbers of **Table 3** are used in (47-48), RHS. Next, we have by (47), (29):

$$\lambda_i\,(t) \;\equiv\; \frac{L_i\,(t)}{L\,(t)} \;=\; \frac{l_i \cdot n_i\,(t)}{l_{15-69}\,(t) \cdot n_{15-69}\,(t)} \quad ; \quad l_i = l_i\,(2010)\;, \quad i = 1,2,..,M \;;\; M = 11$$

$$w_i\,(t) \;=\; \widetilde{w_4}\,(t) \cdot \frac{\alpha_i\,\rho_i}{\alpha_4\,\rho_4}\frac{\lambda_i\,(t)^{\,\rho_i - 1}}{\lambda_4\,(t)^{\,\rho_4 - 1}} \;;\; k = 2.72\,,\;\; p \cdot y = 581972\,,\;\; w = W_A = 437552\; DKK \quad (48)$$

Thus, $\mathbf{w_i(t)}$, $i = 1, 2, ., M$, (48), give all *pecuniary* (money) **wages** in **Tables (5a-5b)**, and/or as exhibited in any/every **Figure 2-6**. As seen in (47-48), $\forall\, t :\; k\,(t),\, y\,(t), W_A\,(t)$, are *unchanged* during projection period 2010-2090 ; $W_A\,(t)$ is the *arithmetic mean* wage rate of the nation's year-round, full-time workers, (46) - **exogenous**, $W_A\,(2010)$, cf. (42).

Around such arithmetic mean, **Macro** wage rate, $\mathbf{W_A\,(2010)}$, however, the '*Division of Labor*' with different **Ages** (maturity/experience) of **workers** amply generate at **Micro** level a *changing* **wage structure** over *time*, given by the *Age wage profiles*, $\mathbf{w_i(t)}$, $\mathbf{i = 1, 2, ., M}$, (48). The *explicit* form (48) shows that the **money age-wage** *determinants* are: 1. [$\mathbf{n_i\,(t)}$, $\mathbf{l_{15-69}\,(t)}$, $\mathbf{n_{15-69}\,(t)}$], by affecting continuously *changing*, $\boldsymbol{\lambda}_\mathbf{i}\,\mathbf{(t)}$, **Age distributions**[26] of demographic induced *Labor supplies*, $L(t) = \sum_{i=1}^{M} L_i(t)$, (39), due to changing *Employment/Population* (Support) ratio, L(t)/N(t), 2. the *endogenous* money **wage**, $\widetilde{\mathbf{w_4}}\,\mathbf{(t)}$, (29), 3. the CRESH **parameters**, $(\alpha_i\,, \rho_i)$ in **Table 4**.

The *Age-wage* **solutions**, $\mathbf{w_i(t)}$, $i = 1, 2, ., M$, (48) in **Tables (5a-5b)**, **Fig. 2-6**, with changing *Support* ratio, L(t)/N(t), may be considered as **Micro** *Age-wage* **scenarios** evolving under **Macro** '**steady-state**' conditions ['steady-state' *sizes* of aggregate capital-labor ratio ($\mathbf{k}$), aggregate labor productivity ($\mathbf{y}$), aggregate wage, $\mathbf{w} = \mathbf{W_A}$].

[26]Edin and Holmlund, (1995, p.328-29) show how marked fluctuations ('shocks') in *calendar* year *sizes* of Swedish (Totals, N_{15-19}) translate into substantial changes in *Age* distribution (*ratios*), $n_i(t)$, $i = 1, ., M$ - coinciding with rising/falling Youth relative wages. Adjusting $n_i\,(t)$ changes $\lambda_i(t)$ in entire *Age* distribution of $L(t)$ - affecting *wage structure*/calendar year *Age wage profiles*, $w_i(t)$, $i = 1., M$, (48).

The **shape** (*qualitative* properties) of the quantitative Age-wage *profiles* (48) will be *robust* and carry over to 'non-steady-state' conditions with *increasing* aggregate Labor productivity ($Y/L = y$) and *increasing* per capita National Income, (46) - as the result of *Capital Accumulation* beyond 'capital widening' to 'capital deepening' [*increasing* capital-labor ratios, $k(t)$] in well-known macro-, two sector [27]-, and multisector growth models. Total (Aggregate) Labor *supply*, $L(t)$, in such growth models could still be the demographic induced *Labor* supplies, $L(t) = \sum_{i=1}^{M} L_i(t)$, (39), that could allow for also generating *Micro* age-wage profiles, $w_i(t)$, $i = 1, 2, ., M$, from quantitative *growth* models.

7 Final Comments and Conclusion

This paper generalized models of imperfect labor substitution/complementarity by simultaneously: (i) specifying the CRESH Labor Aggregator function - relaxing the assumption of single-level, Arrow et al. (1961, p.230), CES elasticity of substitution between labor age groups - dually CES complementarity elasticities of wages to age-group supplies,
(ii) allowing for a much larger number of age groups than is common in the literature,
(iii) CRESH modelling the evolution and consequences of several demographic variants over longer transition periods rather than having a constant age distribution of the population, the labor force, and within and between the cohorts.

We have quantitatively demonstrated the micro-macro economic impacts of the assumptions - alternative fertility scenarios in the demographic projections (2020-2090) - on calendar year (t) wage patterns (Age-wage profiles) in the 'short-run', coming years (decade), and in the 'long-run' upon the lifetime wage incomes for selected (Generations), Cohorts (T), within the period (2010-2035).

The CRESH Labor Aggregator functional form can easily be analytically extended (specified) to include CRESH Subaggregator functions for any relevant Disaggregated Labor categories. Furthermore, the CRESH Labor aggregate (or Labor subaggregates) can next be combined with other Production factors (Capital inputs) in proper specified CRESH Multi-factor Production functions to be applied in single-sector (Macro) or multi-

[27]Equipment investment are among prime determinants to national growth performance (productivity, per capita growth), Jensen (2003, p.82). Machinery becomes "*cheap* as well as *good*," Mokyr (1990, p.87).

sector GE models. In such interaction, the National Aggregate wage, (W_A), become endogenously generated and can provide unified macro equilibrium feedback in calendar years to forming the Age-wage profiles of Labor age-groups and to selected Labor cohorts.

We have come a long way and reached a higher vantage point, which offer a better outlook and apprehension of the roads passed. In closing, we look forward to see Demography and in particular Labor Economics promoting coherent quantifications and projections of real-world (calendar) Annual wages and full-time Employment (Labor years), based on relevant Demographic Register data and consistent with National Income Accounts.

8 App.A : Labor Substitution and Complementarity

8.1 Substitution elasticities and complementarity elasticities

Allen-Uzawa *partial substitution elasticities*, σ_{ij} of any factor pair, (L_i, L_j), for CRESH, (7-8) - Hanoch (1971, p.699), Hanoch (1979, p.296), Guest & Jensen (2016, p.29) - are :

$$\sigma_{ij} = \frac{1}{(1-\rho_i)(1-\rho_j)\bar{\rho}} = \sigma_{ji} > 0 \,,\, i \neq j \,;\; \bar{\rho} = \sum_{i=1}^{M} \frac{\varepsilon_i}{1-\rho_i} \tag{49}$$

$$\sigma_{ii} = \frac{1}{(1-\rho_i)} \left[\frac{1}{(1-\rho_i)\bar{\rho}} - \frac{1}{\varepsilon_i} \right] < 0 \,,\;\; i = 1, ..., M \tag{50}$$

where (σ_{ii}) are the "total substitution elasticity" terms; the *variable* $(\bar{\rho})$ is a weighted average of the parameters, $1/(1-\rho_i)$, with the respective wage (cost) *shares* (ε_i) as variable weights. Clearly, especially *larger* values of ρ_i and ρ_j give a *larger* σ_{ij}. The *restrictions* (8) imply that $\sigma_{ij} > 0$: *all* CRESH labor *inputs* L_i, i=1,...,M, are *substitutes.* If all $\rho_i > 0$, then **all** $\sigma_{ij} > 1$, (49). Note also that **any** σ_{ij} given by (49) via *shares* ε_i, (14), depends on *all* the *parameters*, ρ_i, α_i, and *all* the *Labor inputs* L_i , $i = 1, ..., M$.

It follows from (49) that, although *all* (σ_{ij}) are *variable* elasticities of substitution (**VES**), they have nevertheless an **invariant** (constant) **CRESH** pattern:

$$\frac{\sigma_{ik}}{\sigma_{jk}} = \frac{(1-\rho_j)}{(1-\rho_i)} \,;\; \rho_i > \rho_j : \sigma_{ik} > \sigma_{jk} \;;\; \forall k \neq i, j : \; \frac{\sigma_{ij}}{\sigma_{kl}} = \frac{(1-\rho_k)(1-\rho_l)}{(1-\rho_i)(1-\rho_j)} \tag{51}$$

The *restrictions* (8) and *expressions* (49-51) were obtained by Hanoch (1971, p.698) via Lagrangian cost minimizing *factor demand* functions that correspond to a unique CRESH *minimum Cost* function, $\mathbf{C}(\mathbf{w_1}, \mathbf{w_2}, , , \mathbf{w_M}, \mathbf{L_A})$, or *unit cost* functions, $\mathbf{c}(\mathbf{w_1}, \mathbf{w_2}, , , \mathbf{w_M})$,

$$C(w_1, w_2, , w_M, L_A) = c(w_1, w_2, ., w_M) L_A = \sum_{i=1}^{M} w_i L_i \;;\; L_i = \frac{\partial C}{\partial w_i} \;;\; E(C, w_i) = \varepsilon_i \tag{52}$$

dual to the *implicit* CRESH *production* (aggregator) function, (7-8), (10) , $\varepsilon_i = \frac{w_i L_i}{C}$, (14). The own-price/cross-price *factor demand* elasticities corresponding to (49-50), (52), are:

$$E(L_i, w_i) = \varepsilon_i \sigma_{ii} \;;\; E(L_i, w_j) = \varepsilon_j \sigma_{ij} \;;\; E(L_j, w_i) = \varepsilon_i \sigma_{ji} \;;\quad i = 1, \ldots, M. \tag{53}$$

$E(L_i, w_i)$, $E(L_i, w_j)$ are *conditional* (compensated, *fixed* : L_A) Labor demand elasticities. Like two-factor production/cost functions, the *changes* in factor *shares* (ε_i) are ruled by,

$$\frac{\partial \varepsilon_i}{\partial w_j} \gtreqless 0 \Leftrightarrow \sigma_{ij} \gtreqless 1 \Leftrightarrow E(\varepsilon_i, w_j) = \varepsilon_j (\sigma_{ij} - 1) \;;\; i \neq j \,,\;\; i = 1, ..., M \tag{54}$$

The CRESH elasticities, (53) and (49-50), satisfy the standard summation properties :

$$\sum_{j=1}^{M} E\,(L_i, w_j) \;=\; \sum_{j=1}^{M} \varepsilon_j\, \sigma_{ji} \;=\; 0 \;\;;\;\; \sum_{j=1}^{M} \varepsilon_j\, E\,(L_j, w_i) \;=\; 0 \tag{55}$$

Actual *parametric* CRESH substitution elasticities, σ_{ij}, σ_{ii}, (49-50), and factor demand elasticities, $E\,(L_i, w_i)$, $E\,(L_i, w_j)$, (53), (55), are shown in **Table 8A-8B**, using the validated CRESH *parameter* values : $(\rho_i,\, \alpha_i)$ in **Table 4**, (column 3-4), and the *Labor inputs* L_i, i=1,..,M, $L_A = L$, (2010), in **Table 2** (column 5).

The σ_{ij} formulas (56) are the Uzawa (1962, p.293) *duality forms* of Allen (1938, p.504)

$$\sigma_{ij} \;=\; \frac{C\,\frac{\partial^2 C}{\partial w_i \partial w_j}}{\frac{\partial C}{\partial w_i}\frac{\partial C}{\partial w_j}} \;\;,\;\; \sigma_{ii} \;=\; \frac{C\,\frac{\partial^2 C}{\partial w_i^2}}{[\frac{\partial C}{\partial w_i}]^2} \;\;; \qquad c_{ij} \;=\; \frac{f\,\frac{\partial^2 f}{\partial L_i \partial L_j}}{\frac{\partial f}{\partial L_i}\frac{\partial f}{\partial L_j}} \;\;,\;\; c_{ii} \;=\; \frac{f\,\frac{\partial^2 f}{\partial L_i^2}}{[\frac{\partial f}{\partial L_i}]^2} \tag{56}$$

partial elasticity of substitution (σ_{ij}). But it is *impossible* to *apply* the beautiful and *simple* σ_{ij} formulas (56) to get the CRESH results (49-50), as the relevant **dual** CRESH *cost function*, $C\,(w_1, w_2, , , w_M, L_A)$, (52), has *no closed* form. However, such existing unknown *dual* CRESH *Cost* function (52) would by σ_{ij} (56) give the same CRESH *parametric substitution elasticities* (49-50) - as were successfully derived from the first and second order conditions for CRESH Lagrangian cost minimization by Hanoch (1971, p.697-98).

The Hicks *partial complementarity elasticity* (c_{ij}), (56), for any factor pair (L_i, L_j) of CRESH function, $L_A = f\,(L_1, L_2, ., L_M)$, (10), are defined exactly in *analogy* with σ_{ij} of C, (52), see Sato and Koizumi (1973, p.47)[28] ; cf. Hicks (1970).[29] Note that the *size* of L_A [level of output, Y (note 35)] is *not* held *constant* in complementarity elasticities, c_{ij}. In fact, *positive* c_{ij} measures exactly the degree to which two factor inputs *jointly* contribute to a *change* in L_A $[Y]$ - as the cross-partial derivative $\frac{\partial^2 f}{\partial L_i \partial L_j}$ shows in (56). Thus in contrast to σ_{ij} , (49), *larger* values of ρ_i and ρ_j give *smaller* numbers for c_{ij} , (57).

[28] Sato & Koizumi (1973, p.46) considered an *explicit* production function as, $Y = F(X_1, X_2, \ldots, X_M)$, Y = output, X_i = i-th input, with derivatives , $\forall X_i > 0$: $\frac{\partial F}{\partial X_i} > 0$, $\frac{\partial^2 F}{\partial X_i^2} < 0$, $Y = \sum_{i=1}^{M} \frac{\partial F}{\partial X_i} X_i$.

The *complementarity elasticities*, c_{ij} , c_{ii}, are defined as : $c_{ij} \;=\; \frac{F\,\frac{\partial^2 F}{\partial X_i \partial X_j}}{\frac{\partial F}{\partial X_i}\frac{\partial F}{\partial X_j}}$, $c_{ii} \;=\; \frac{F\,\frac{\partial^2 F}{\partial X_i^2}}{[\frac{\partial F}{\partial X_i}]^2}$

The problem with applying these c_{ij} , c_{ii}, definitions to CRESH function, $L_A = f\,(L_1, L_2, ., L_M)$, above in (56) is that f, (10), is *not* - as here $F(X_1, X_2, \ldots, X_M)$ - an *explicit* function. However, we *know* (can calculate, as explained in section 8.2 and **Appendix B**) the *derivatives* of f, (10), to use in (56), and $L_A = f$ drops out of (56) - as seen from CRESH formulas (72-73), (78-79), and finally stated in (57-58).

[29] On related issues of Derived Factor Demand, see Sato & Koizumi (1970, p.109), Hicks (1970, p.294).

Table 8A. Partial substitution elasticities $(\sigma_{ij}, \sigma_{ii})$ - Denmark 2010 - by (49-50) and Table 4.

	L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8	L_9	L_{10}	L_{11}
L_1	-809.675	8.679	5.786	5.786	4.340	3.472	3.472	4.340	5.786	8.679	8.679
L_2	8.679	-144.081	5.786	5.786	4.340	3.472	3.472	4.340	5.786	8.679	8.679
L_3	5.786	5.786	-47.513	3.858	2.893	2.315	2.315	2.893	3.858	5.786	5.786
L_4	5.786	5.786	3.858	-26.883	2.893	2.315	2.315	2.893	3.858	5.786	5.786
L_5	4.340	4.340	2.893	2.893	-15.561	1.736	1.736	2.170	2.893	4.340	4.340
L_6	3.472	3.472	2.315	2.315	1.736	-11.409	1.389	1.736	2.315	3.472	3.472
L_7	3.472	3.472	2.315	2.315	1.736	1.389	-11.530	1.736	2.315	3.472	3.472
L_8	4.340	4.340	2.893	2.893	2.170	1.736	1.736	-16.036	2.893	4.340	4.340
L_9	5.786	5.786	3.858	3.858	2.893	2.315	2.315	2.893	-23.6932	5.786	5.786
L_{10}	8.679	8.679	5.786	5.786	4.340	3.472	3.472	4.340	5.786	-67.354	8.679
L_{11}	8.679	8.679	5.786	5.786	4.340	3.472	3.472	4.340	5.786	8.679	-419.695
$\sum_{j=1}^{M} \varepsilon_j \sigma_{ji}$	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Table 8B. Conditional factor demand elasticities, $E(L_i, w_i), E(L_i, w_j)$ - by (53) and Table 4.

	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8	w_9	w_{10}	w_{11}	$\sum_{i=1}^{M} E(L_i, w_j)$
L_1	-4.947	0.284	0.375	0.627	0.612	0.543	0.537	0.596	0.700	0.571	0.101	0.000
L_2	0.053	-4.716	0.375	0.627	0.612	0.543	0.537	0.596	0.700	0.571	0.101	0.000
L_3	0.035	0.189	-3.083	0.418	0.408	0.362	0.358	0.397	0.467	0.381	0.068	0.000
L_4	0.035	0.189	0.250	-2.915	0.408	0.362	0.358	0.397	0.467	0.381	0.068	0.000
L_5	0.027	0.142	0.188	0.314	-2.194	0.271	0.269	0.298	0.350	0.285	0.051	0.000
L_6	0.021	0.114	0.150	0.251	0.245	-1.783	0.215	0.238	0.280	0.228	0.041	0.000
L_7	0.021	0.114	0.150	0.251	0.245	0.217	-1.785	0.238	0.280	0.228	0.041	0.000
L_8	0.027	0.142	0.188	0.314	0.306	0.271	0.269	-2.202	0.350	0.285	0.051	0.000
L_9	0.035	0.189	0.250	0.418	0.408	0.362	0.358	0.397	-2.867	0.381	0.068	0.000
L_{10}	0.053	0.284	0.375	0.627	0.612	0.543	0.537	0.596	0.700	-4.429	0.101	0.000
L_{11}	0.053	0.284	0.375	0.627	0.612	0.543	0.537	0.596	0.700	0.571	-4.899	0.000
$\sum_{j=1}^{M} \varepsilon_j E(L_j, w_i)$	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	

8.2 Labor Complementarity elasticities

The elasticity c_{ij} *formulas* (56) are simple ; but the CRESH, $\mathbf{L_A} = \mathbf{f}\,(\mathbf{L_1}, \mathbf{L_2}, ..., \mathbf{L_M})$, (10), did *not* exist in *closed* form. For CRESH, (7-8), the *complementarity elasticities* c_{ij}, (56), are *explicitly* derived in **Appendix B**, (78-79), as *parametrically* given by :

$$c_{ij} \;=\; 1 - \rho_i - \rho_j + \tilde{\rho} \;=\; c_{ji}\;,\;\; i \neq j\;;\;\; \tilde{\rho} \;=\; \sum_{i=1}^{M} \varepsilon_i\, \rho_i \qquad (57)$$

$$c_{ii} \;=\; 1 - 2\rho_i \;-\; (1 - \rho_i)/\varepsilon_i \;+\; \tilde{\rho} \;<\; 0\,,\;\; i = 1, ..., M \qquad (58)$$

where (c_{ii}) are the "total complementarity elasticity" terms ; *variable* ($\tilde{\rho}$) is a weighted average of parameters (ρ_i), with the respective wage (cost) *shares* (ε_i) as variable weights. **CRESH** (c_{ij}) are all *variable complementarity* elasticities, (57), but they have an **invariant** (constant) **CDEC** (*constant difference* of *elasticity* of *complementarity*) pattern:

$$c_{ik} - c_{jk} \;=\; \rho_i - \rho_j\;; \quad c_{ij} - c_{kl} \;=\; (\rho_i + \rho_j) - (\rho_k + \rho_l) \qquad (59)$$

Note that *unlike* substitution elasticities, σ_{ij}, (49), the *restrictions* (8) do **not** impose a particular *sign* upon **all** the **complementarity** elasticities, c_{ij}, (57).

Wage Income function, $\mathbf{W} = \mathbf{W}\,(\mathbf{L_1}, \mathbf{L_2}, ..., \mathbf{L_M}, \mathbf{W_A})$ - as a **dual** to *Wage Cost* function, $\mathbf{C}(\mathbf{w_1}, \mathbf{w_2}, , , \mathbf{w_M}, \mathbf{L_A})$, (52) - is an *alternative Wage Sum* formulation with *important applications* in sections 4-5 for the *elasticities* c_{ij}, (57), see **Appendix B**, (81-88):

$$W\,(L_1, L_2, .., L_M, W_A) = W_A\, f\,(L_1, L_2, .., L_M) = W_A L_A\,;\; w_i = \frac{\partial W}{\partial L_i},\; E(W, L_i) = \varepsilon_i \quad (60)$$

$$W(L_1, L_2, , , L_M, W_A) \;=\; c\,(w_1, w_2, , , w_M)L_A = W_A L_A \;=\; \sum_{i=1}^{M} w_i L_i\;;\;\; W_A \;=\; c\,(w_1, w_2, , , w_M) \qquad (61)$$

$$c_{ij} \;=\; c_{ij}\,(L_1, L_2, ..., L_M) \;=\; \frac{W\,\frac{\partial^2 W}{\partial L_i \partial L_j}}{\frac{\partial W}{\partial L_i}\,\frac{\partial W}{\partial L_j}} \;\;;\;\; i = 1, ..., M\,,\;\; j = 1, ..., M\;; \quad c_{ii} \;=\; c_{ii}\,(L_1, L_2, ..., L_M) \;=\; \frac{W\,\frac{\partial^2 W}{\partial L_i^2}}{[\frac{\partial W}{\partial L_i}]^2} \qquad (62)$$

Factor price (**wage**) elasticities w.r.t *own* -, *cross* **supply** *increases* are, cf. (53), (57-58),

$$E\,(w_i, L_i) \;=\; \varepsilon_i\, c_{ii}\;,\;\; E\,(w_i, L_j) \;=\; \varepsilon_j\, c_{ij}\;\;,\;\; E\,(w_j, L_i) \;=\; \varepsilon_i\, c_{ji}\;\;; \quad i = 1, \ldots, M. \qquad (63)$$

$E\,(w_i, L_i)$, $E\,(w_i, L_j)$, are *conditional* (fixed $\mathbf{W_A}$) **partial** *wage* elasticities of *group (i).* Like two-factor production/cost functions, *wage* **shares** (ε_i), (14), (60), follow the rules:

$$\frac{\partial \varepsilon_i}{\partial L_j} \;\gtreqless\; 0 \;\Leftrightarrow\; c_{ij} \;\gtreqless\; 1 \;\Leftrightarrow\; E\,(\varepsilon_i, L_j) \;=\; \varepsilon_j\,(c_{ij} - 1)\;; \quad i \neq j\,,\;\; i = 1, ..., M \qquad (64)$$

Note that $\mathbf{c_{ij}}$ in (57) depend on *all parameters*, ρ_i, α_i, and also via : ε_i, $(\tilde{\rho})$, on *all* the *Labor inputs*, $L_i, i = 1, ..., M$, cf. (14), (49). Thereby is $\mathbf{c_{ij}}$ the relevant and adequate *tool* (summary measure) - with our CRESH forms, (57) - to answer *distributional* (absolute *wage share*) issues with formula (64) ; cf. (54). See Sato and Koizumi (1973, p.486). CRESH elasticities, (62-63), (57-58), (89), have standard *summation* properties, cf. (55):

$$\sum_{j=1}^{M} E(w_i, L_j) = \sum_{j=1}^{M} \varepsilon_j c_{ji} = 0 \;\; ; \;\; \sum_{j=1}^{M} \varepsilon_j E(w_j, L_i) = 0 \tag{65}$$

Actual *parametric* CRESH *complementarity* elasticities, c_{ij}, c_{ii}, (57-58), and the *wage effect* elasticities, $E(w_i, L_i)$, $E(w_i, L_j)$, (63), (65), are shown in **Tables 8C-8D**, cf. CRESH *parameter* values : $(\rho_i, \ \alpha_i)$ in **Table 4**, (column 3-4), and the *Labor inputs* L_i, i=1,..,M, $L_A = L$, (2010), in **Table 2** (column 5).

Note in **Table 8D** that the numerically *highest* **wage** elasticities, $E(w_i, L_i)$, are : $E(w_5, L_5)$, $E(w_6, L_6)$, $E(w_7, L_7)$, $E(w_8, L_8)$, i.e., being most *sensitive* to **own** *supply* increases. These same *middle age* groups *gain* most by *larger* **cross** *supplies* from other age-groups, i.e., have the *highest* **wage** elasticities, $E(w_i, L_j)$, $i = 5, 6, 7, 8$, $i \neq j$, in **Table 8D** - they have also the *largest* cross *complementarity* elasticities in **Table 8C**.

Finally, let us note from (49) and (57), that **if** $\forall \rho_i = \rho$, cf. CES, (9), then we have,

$$\forall \rho_i = \rho : \qquad \sigma_{ij} = \frac{1}{(1-\rho_i)(1-\rho_j)\,\bar{\rho}} = \frac{1}{1-\rho}\,; \qquad c_{ij} = 1 - \rho_i - \rho_j + \tilde{\rho} = 1 - \rho \tag{66}$$

i.e., substitution elasticities (σ_{ij}) and *dual* complementarity elasticities (c_{ij}) are simply *reciprocals* of each other, and there would also be simple "reciprocal" relations between *factor demand* elasticities (53) and the so-called "*inverse factor* demand" [conditional *partial* **wage**] elasticities, (63). But with the much *richer parametric* class of **CRESH** production/aggregator functions and their duality relations, the *simple reciprocals* in (66) evidently **no longer** *apply* - and clearly **Table 8C** is *neither* the *reciprocal* of **Table 8A**.

With *demographic Age* groups and *exogenous* Labor *supplies* (L_i), (39), it is $\mathbf{c_{ij}}$, (57-58), and $E(w_i, L_i)$, $E(w_i, L_j)$, $E(w_j, L_i)$, $E(w_i, L_j)$, (63-65), **Tables 8C-8D** that are the *relevant* elasticities - which are behind all the *Age-wage group* results in **Tables 5-7**.

Table 8C. Partial complementarity elasticities (c_{ij}, c_{ii}) - Denmark 2010 - by (57-58) and Table 4.

	L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8	L_9	L_{10}	L_{11}
L_1	-32.713	0.022	0.122	0.122	0.222	0.322	0.322	0.222	0.122	0.022	0.022
L_2	0.022	-6.089	0.122	0.122	0.222	0.322	0.322	0.222	0.122	0.022	0.022
L_3	0.122	0.122	-4.402	0.222	0.322	0.422	0.422	0.322	0.222	0.122	0.122
L_4	0.122	0.122	0.222	-2.545	0.322	0.422	0.422	0.322	0.222	0.122	0.122
L_5	0.222	0.222	0.322	0.322	-2.415	0.522	0.522	0.422	0.322	0.222	0.222
L_6	0.322	0.322	0.422	0.422	0.522	-2.578	0.622	0.522	0.422	0.322	0.322
L_7	0.322	0.322	0.422	0.422	0.522	0.622	-2.608	0.522	0.422	0.322	0.322
L_8	0.222	0.222	0.322	0.322	0.422	0.522	0.522	-2.491	0.322	0.222	0.222
L_9	0.122	0.122	0.222	0.222	0.322	0.422	0.422	0.322	-2.258	0.122	0.122
L_{10}	0.022	0.022	0.122	0.122	0.222	0.322	0.322	0.222	0.122	-3.020	0.022
L_{11}	0.022	0.022	0.122	0.122	0.222	0.322	0.322	0.222	0.122	0.022	-17.113
$\sum_{j=1}^{M} \varepsilon_j c_{ji}$	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Table 8D. Partial factor price (wage) elasticities, $E(w_i, L_i), E(w_i, L_j)$ by (63) and Table 4.

	L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8	L_9	L_{10}	L_{11}	$\sum_{j=1}^{M} E(w_i, L_j)$
w_1	-0.200	0.001	0.008	0.013	0.031	0.050	0.050	0.030	0.015	0.001	0.000	0.000
w_2	0.000	-0.199	0.008	0.013	0.031	0.050	0.050	0.030	0.015	0.001	0.000	0.000
w_3	0.001	0.004	-0.286	0.024	0.045	0.066	0.065	0.044	0.027	0.008	0.001	0.000
w_4	0.001	0.004	0.014	-0.276	0.045	0.066	0.065	0.044	0.027	0.008	0.001	0.000
w_5	0.001	0.007	0.021	0.035	-0.341	0.082	0.081	0.058	0.039	0.015	0.003	0.000
w_6	0.002	0.011	0.027	0.046	0.074	-0.403	0.096	0.072	0.051	0.021	0.004	0.000
w_7	0.002	0.011	0.027	0.046	0.074	0.097	-0.404	0.072	0.051	0.021	0.004	0.000
w_8	0.001	0.007	0.021	0.035	0.059	0.082	0.081	-0.342	0.039	0.015	0.003	0.000
w_9	0.001	0.004	0.014	0.024	0.045	0.066	0.065	0.044	-0.273	0.008	0.001	0.000
w_{10}	0.000	0.001	0.008	0.013	0.031	0.050	0.050	0.030	0.015	-0.199	0.000	0.000
w_{11}	0.000	0.001	0.008	0.013	0.031	0.050	0.050	0.030	0.015	0.001	-0.200	0.000
$\sum_{j=1}^{M} \varepsilon_j E(w_j, L_i)$	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	

9 Appendix B : CRESH complementarity elasticities

A1. The *partial complementarity elasticity*, (c_{ij}), between any factor pair (L_i, L_j) within in the *implicit* CRESH function, $\mathbf{L_A} = \mathbf{f}(\mathbf{L_1}, \mathbf{L_2}, ..., \mathbf{L_M})$, (10), was defined in (56) as :

$$c_{ij} \equiv c_{ij}(L_1, L_2, ..., L_M) = \frac{f\,\frac{\partial^2 f}{\partial L_i \partial L_j}}{\frac{\partial f}{\partial L_i}\frac{\partial f}{\partial L_j}} \;;\; i = 1, ..., M\,,\; j = 1, ..., M\,; \quad c_{ii} \equiv c_{ii}(L_1, L_2, ..., L_M) = \frac{f\,\frac{\partial^2 f}{\partial L_i^2}}{[\frac{\partial f}{\partial L_i}]^2} \tag{67}$$

The *first-order* derivatives in (67) were already given in (12) as,

$$\forall L_i > 0 :\; \frac{\partial f}{\partial L_i} = -\frac{\partial F/\partial L_i}{\partial F/\partial L_A} = \frac{\alpha_i \rho_i (L_i/L_A)^{\rho_i - 1}}{\sum_{i=1}^{M} \alpha_i \rho_i (L_i/L_A)^{\rho_i}} > 0\,, \qquad i = 1, ..., M \tag{68}$$

The *second-order* derivatives in (67) are derived from the second term (ratio) in (68) as,

$$\frac{\partial^2 f}{\partial L_i \partial L_j} = \frac{-1}{[\frac{\partial F}{\partial L_A}]^3}\left[\frac{\partial^2 F}{\partial L_i \partial L_j}\left[\frac{\partial F}{\partial L_A}\right]^2 - \frac{\partial^2 F}{\partial L_i \partial L_A}\frac{\partial F}{\partial L_j}\frac{\partial F}{\partial L_A} - \frac{\partial^2 F}{\partial L_j \partial L_A}\frac{\partial F}{\partial L_i}\frac{\partial F}{\partial L_A} + \frac{\partial^2 F}{\partial L_A^2}\frac{\partial F}{\partial L_i}\frac{\partial F}{\partial L_j}\right] \tag{69}$$

$$\frac{\partial^2 f}{\partial L_i^2} = \frac{-1}{[\frac{\partial F}{\partial L_A}]^3}\left[\frac{\partial^2 F}{\partial L_i^2}\left[\frac{\partial F}{\partial L_A}\right]^2 - 2\,\frac{\partial^2 F}{\partial L_i \partial L_A}\frac{\partial F}{\partial L_i}\frac{\partial F}{\partial L_A} + \frac{\partial^2 F}{\partial L_A^2}\left[\frac{\partial F}{\partial L_i}\right]^2\right] \tag{70}$$

Insert first-order and second-order derivatives (68-70) of $L_A = f(L_1, L_2, .., L_M)$ into (67):

$$c_{ij} = -L_A\left[\frac{\frac{\partial^2 F}{\partial L_i \partial L_j}\frac{\partial F}{\partial L_A}}{\frac{\partial F}{\partial L_i}\frac{\partial F}{\partial L_j}} - \frac{\frac{\partial^2 F}{\partial L_i \partial L_A}}{\frac{\partial F}{\partial L_i}} - \frac{\frac{\partial^2 F}{\partial L_j \partial L_A}}{\frac{\partial F}{\partial L_j}} + \frac{\frac{\partial^2 F}{\partial L_A^2}}{\frac{\partial F}{\partial L_A}}\right] \tag{71}$$

In CRESH cases, we have, $\partial^2 F/\partial L_i \partial L_j = 0$, if $i \neq j$, cf. (74-75). Hence (71) becomes :

$$c_{ij} = -L_A\left[-\frac{\frac{\partial^2 F}{\partial L_i \partial L_A}}{\frac{\partial F}{\partial L_i}} - \frac{\frac{\partial^2 F}{\partial L_j \partial L_A}}{\frac{\partial F}{\partial L_j}} + \frac{\frac{\partial^2 F}{\partial L_A^2}}{\frac{\partial F}{\partial L_A}}\right]\,, \quad i \neq j \tag{72}$$

$$c_{ii} = -L_A\left[\frac{\frac{\partial^2 F}{\partial L_i^2}\frac{\partial F}{\partial L_A}}{[\frac{\partial F}{\partial L_i}]^2} - 2\,\frac{\frac{\partial^2 F}{\partial L_i \partial L_A}}{\frac{\partial F}{\partial L_i}} + \frac{\frac{\partial^2 F}{\partial L_A^2}}{\frac{\partial F}{\partial L_A}}\right]\,, \quad i = j \tag{73}$$

To obtain *explicit* CRESH formulas from (72-73), the *parametric* expressions of the first-order and second-order derivatives of the CRESH function, $F(L_A, L_1, L_2, ..., L_M)$, (7) are now needed. We already have the first-order derivatives of F as, cf. (11),

$$\frac{\partial F}{\partial L_i} = \frac{\gamma \alpha_i \rho_i (L_i/L_A)^{\rho_i - 1}}{L_A} \equiv \frac{\gamma\, \varepsilon_i\, \beta}{L_i}\;;\quad \frac{\partial F}{\partial L_A} = -\frac{\gamma \sum_{i=1}^{M} \alpha_i \rho_i (L_i/L_A)^{\rho_i}}{L_A} \equiv -\frac{\gamma\, \beta}{L_A} \tag{74}$$

where, $\beta \equiv \sum_{\mathbf{i=1}}^{\mathbf{M}} \alpha_{\mathbf{i}} \rho_{\mathbf{i}} (\mathbf{L_i}/\mathbf{L_A})^{\rho_{\mathbf{i}}}$; $\varepsilon_i = \alpha_{\mathbf{i}} \rho_{\mathbf{i}} (\mathbf{L_i}/\mathbf{L_A})^{\rho_{\mathbf{i}}}/\beta$, i=1,...,M, cf. (14).

The second-order derivatives of F are derived from the second terms (ratios) in (74) as,

$$\frac{\partial^2 F}{\partial L_i \partial L_j} = 0;\; \frac{\partial^2 F}{\partial L_i \partial L_A} = -\gamma \alpha_i \rho_i^2 L_i^{\rho_i - 1} L_A^{-1-\rho_i} = -\frac{\gamma\, \rho_i\, \varepsilon_i\, \beta}{L_i L_A}\,;\; \frac{\partial^2 F}{\partial L_j \partial L_A} = -\frac{\gamma\, \rho_j\, \varepsilon_j\, \beta}{L_j L_A} \tag{75}$$

$$\frac{\partial^2 F}{\partial L_i^2} = \frac{1}{L_A}\gamma\alpha_i\rho_i(\rho_i-1)\,(L_i/L_A)^{\rho_i-2}\,\frac{1}{L_A} = \frac{\gamma\,(\rho_i-1)\,\varepsilon_i\,\beta}{L_i^2} \tag{76}$$

$$\frac{\partial^2 F}{\partial L_A^2} = \gamma\sum_{i=1}^{M}\alpha_i\rho_i(1+\rho_i)L_i^{\rho_i}L_A^{-2-\rho_i} = \frac{\gamma\,\beta}{L_A^2}\,[\,\sum_{i=1}^{M}(1+\rho_i)\,\varepsilon_i\,] = \frac{\gamma\,\beta}{L_A^2}\,(1+\tilde{\rho}\,) \tag{77}$$

where, $\tilde{\rho} = \sum_{\mathbf{i=1}}^{\mathbf{M}} \varepsilon_{\mathbf{i}}\,\rho_i$. Finally, inserting (74-77) into (72-73) give,

$$c_{ij} = -\,L_A\left[-\frac{\frac{-\gamma\,\rho_i\,\varepsilon_i\,\beta}{L_iL_A}}{\frac{\gamma\,\varepsilon_i\,\beta}{L_i}} - \frac{\frac{-\gamma\,\rho_j\,\varepsilon_j\,\beta}{L_jL_A}}{\frac{\gamma\,\varepsilon_j\,\beta}{L_j}} + \frac{\frac{\gamma\beta}{L_A^2}\,(1+\tilde{\rho}\,)}{-\frac{\gamma\beta}{L_A}}\right] = -\rho_i-\rho_j+1+\tilde{\rho}\;;\;\tilde{\rho} = \sum_{i=1}^{M}\varepsilon_i\,\rho_i \tag{78}$$

$$c_{ii} = -\,L_A\left[\frac{\frac{\gamma\,(\rho_i-1)\,\varepsilon_i\,\beta}{L_i^2}[-\frac{\gamma\,\beta}{L_A}]}{[\frac{\gamma\,\varepsilon_i\,\beta}{L_i}]^2} - 2\frac{\frac{-\gamma\,\rho_i\,\varepsilon_i\,\beta}{L_iL_A}}{\frac{\gamma\,\varepsilon_i\,\beta}{L_i}} + \frac{\frac{\gamma\beta}{L_A^2}\,(1+\tilde{\rho}\,)}{-\frac{\gamma\beta}{L_A}}\right] = \frac{\rho_i-1}{\varepsilon_i} - 2\rho_i + 1 + \tilde{\rho} \tag{79}$$

Labor *complementarity elasticities* c_{ij} (78-79) satisfy *regularity* (summation) property :

$$\begin{aligned}\sum_{j=1}^{M}\varepsilon_j c_{ji} &= \sum_{j=1,j\neq i}^{M}\varepsilon_j c_{ji} + \varepsilon_i c_{ii} = \sum_{j=1,j\neq i}^{M}\varepsilon_j(1-\rho_i-\rho_j+\tilde{\rho}) + \varepsilon_i(1-2\rho_i+\frac{\rho_i-1}{\varepsilon_i}+\tilde{\rho}) \\ &= -\sum_{j\neq i}^{M}\varepsilon_j\rho_j - \rho_i\sum_{j\neq i}^{M}\varepsilon_j - 2\rho_i\varepsilon_i + \rho_i + \tilde{\rho} = -\sum_{j=1}^{M}\varepsilon_j\rho_j + \tilde{\rho} = -\tilde{\rho}+\tilde{\rho} = 0\end{aligned} \tag{80}$$

CRESH complementarity elasticities (78-80) were seen in **Table 8C** for Denmark (2010).

A2. **Wage Income** function, $\mathbf{W}\,(\mathbf{L_1},\mathbf{L_2},.,\mathbf{L_M},\mathbf{W_A})$ - Wage Sum, W_AL_A, defined as,

$$W\,(L_1,L_2,.,L_M,W_A) = W_A\,f\,(L_1,L_2,..,L_M) = c\,(w_1,w_2,,,w_M)L_A = \sum_{i=1}^{M}w_iL_i \equiv W_AL_A \tag{81}$$

From *Wage Income* function (81), we get the *basic dual* expressions, cf. (14), (52), (67),

$$E(W,L_i) = \frac{\partial W}{\partial L_i}\frac{L_i}{W} = \frac{w_iL_i}{W} = \frac{\partial f}{\partial L_i}\frac{L_i}{f} = E(W,L_i) = \varepsilon_i = E(L_A,L_i)\,;\;E(W,W_A) = 1 \tag{82}$$

$$\frac{W\,\frac{\partial^2 W}{\partial L_i\partial L_j}}{\frac{\partial W}{\partial L_i}\frac{\partial W}{\partial L_j}} = \frac{W_A\,f\;W_A\,\frac{\partial^2 f}{\partial L_i\partial L_j}}{W_A\,\frac{\partial f}{\partial L_i}\;W_A\,\frac{\partial f}{\partial L_j}} = \frac{f\,\frac{\partial^2 f}{\partial L_i\partial L_j}}{\frac{\partial f}{\partial L_i}\frac{\partial f}{\partial L_j}} = c_{ij} = c_{ji}\;\;;\;\;i=1,...,M\,,\;j=1,...,M \tag{83}$$

$$w_i = w_i\,(L_1,L_2,...,L_M,W_A) = \frac{\partial W\,(L_1,L_2,...,L_M,W_A)}{\partial L_i} = W_A\cdot\frac{\partial f\,(L_1,L_2,...,L_M)}{\partial L_i} \tag{84}$$

where w_i (84) is the *"shadow value"* (*marginal value-added* : $W_A\cdot\frac{\partial f}{\partial L_i}$) of *one unit* increase of specific Labor *inputs* from *age*-group (i), $\mathbf{L_i}$, i.e., $\mathbf{w_i}$ is the *nominal factor price* (money annual wage) of $\mathbf{L_i}$ - being obtained as **Inverse** *factor demand* price or named '*partial market equilibrium*' **wage** for the *Labor* **supply** of *Age* (i), $\mathbf{L_i}$ - with *fixed* Labor *supplies* of all the other Age groups - and with a *fixed aggregate* (average) annual *wage*, $\mathbf{W_A}$, to the *Total* (Aggregate) *Labor* market *equilibrium* [complying with full general equilibrium of competitive product and factor markets].

Finally, using (82-83), we shall derive the annual *partial wage* **elasticities** of the optimal (Pareto efficient) **annual** wages (w_i), [*Inverse* factor demands], (84), with respect to *partial* variation of **own** labor *supply* (L_i) and any **cross** labor *supply* (L_j), i.e.,

$$\frac{\partial w_i\,(L_1, L_2, ..., L_M, W_A)}{\partial L_j} = \frac{\partial^2 W\,(L_1, L_2, ..., L_M, W_A)}{\partial L_i\,\partial L_j} \;\; ; \quad i = 1, ..., M \; , \; j = 1, ..., M \quad (85)$$

$$E(w_i, L_j) = \frac{\partial w_i}{\partial L_j}\frac{L_j}{w_i} = \frac{\partial^2 W}{\partial L_i\,\partial L_j}\frac{L_j}{w_i} = \frac{\frac{\partial^2 W}{\partial L_i\,\partial L_j}\,L_j}{\frac{\partial W}{\partial L_i}} = \frac{W\,\frac{\partial^2 W}{\partial L_i\,\partial L_j}\cdot\frac{w_j L_j}{W}}{\frac{\partial W}{\partial L_i}\frac{\partial W}{\partial L_j}} = \varepsilon_j\,c_{ij} \quad (86)$$

$$E(w_j, L_i) = \frac{\partial w_j}{\partial L_i}\frac{L_i}{w_j} = \frac{\partial^2 W}{\partial L_j\,\partial L_i}\frac{L_i}{w_j} = \frac{\frac{\partial^2 W}{\partial L_j\,\partial L_i}\,L_i}{\frac{\partial W}{\partial L_j}} = \frac{W\,\frac{\partial^2 W}{\partial L_j\,\partial L_i}\cdot\frac{w_i L_i}{W}}{\frac{\partial W}{\partial L_j}\frac{\partial W}{\partial L_i}} = \varepsilon_i\,c_{ij} \quad (87)$$

$$E(w_i, L_i) \;=\; \frac{\partial w_i}{\partial L_i}\frac{L_i}{w_i} \;=\; \frac{\partial^2 W}{\partial L_i^2}\frac{L_i}{w_i} \;=\; \frac{\frac{\partial^2 W}{\partial L_i^2}\,L_i}{\frac{\partial W}{\partial L_i}} \;=\; \frac{W\,\frac{\partial^2 W}{\partial L_i^2}\cdot\frac{w_i L_i}{W}}{[\frac{\partial W}{\partial L_i}]^2} \;=\; \varepsilon_i\,c_{ii} \quad (88)$$

where *complementarity elasticities*, c_{ij}, (78-79), are the relevant numbers for obtaining the **basic partial wage elasticities**, (63), (86-88), that are involved in our demographic *population* (cohort) *impact* analyses (calculations) over the projection period, 2020-2090. *Annual wage elasticities* of Labor *supply* (63),(86-88) have *summation* properties, as (65):

$$(i) \quad \sum_{j=1}^{M} E\,(w_i, L_j) \;=\; \sum_{j=1}^{M} \varepsilon_j\,c_{ji} \;=\; 0 \;\; ; \quad (ii) \quad \sum_{j=1}^{M} \varepsilon_j\,E\,(w_j, L_i) \;=\; 0 \quad (89)$$

Annual wage elasticities (63), (86-89), were illustrated in **Table 8D** for *Denmark* (2010).

By the way, the "*adding-up*", summing-property $(89, i)$ is easily understood to hold from the "shadow-value" (wage) functions (84) being homogeneous of degree *zero* in increasing *all* labor supplies - by *derivatives* of the Wage Income function and Aggregator function, (81), (being homogeneous of degree *one* in labor supplies). Increasing proportionally *all* labour supplies does *not* change the *relative wages* - hence economically $(89, i)$. Actual checking $(89, ii)$ for CRESH (80) was more cumbersome ; but this was of course *necessary* for the CRESH formula **demographic-labor applications** in sections 4-5.

9.1 Price functions, Inverse demands - Hotelling-Wold identity

9.1.1. *Existence of consumer good price functions.* With regular (monotone, quasiconcave, smooth) **Utility** functions, $u = U\,(q_1, .., q_n)$, and *Budget constraint*, $P_1 q_1 +, .., +P_n q_n = C$, there is one and *only one set* of consumer good *prices*, P_i, $i = 1, 2, ..., n$, for which exogenously *fixed quantities*, $(q_1, ..., q_n)$, are *optimal* (max.U) ; this price set is given by :

$$\frac{P_i}{C} = \varphi_i\,(q_1, ..., q_n) \equiv \frac{\frac{\partial U}{\partial q_i}(q_1, ..., q_n)}{\frac{\partial U}{\partial q_1}\cdot q_1 + \frac{\partial U}{\partial q_2}\cdot q_2 + ... + \frac{\partial U}{\partial q_n}\cdot q_n}\ ;\ P_i = \varphi_i\, C,\ \ i = 1, 2, ..., n \quad (90)$$

P_i - *price functions*, named as *Hotelling-Wold identity* - shown in Hotelling (1935, p.71)[30], Wold (1944, p.70), Wold & Juren (1953, p.92,p.145) - or **Inverse** *uncompensated consumer good* ('Marshall') *demand* functions, cf. Diewert (1974, p.131), Cornes (1992, p.37). Given explicit, $u = U\,(q_1, ..., q_n)$, the functions, $\varphi_i\,(q_1, ..., q_n)$, (90), are easily obtained.

9.1.2. *Existence of factor price functions.* Given a regular (monotone, concave, smooth, homogeneous of degree one) **Production** function, $Y = g\,(x_1, ..., x_m)$, generating *Total revenue* (Factor income, Value-added), $V \equiv P\,Y = w_1 x_1 +, ..., + w_m x_m$, there is one and *only one set* of *factor prices*, w_i, $i = 1, 2, ..., n$, for which the factor *quantities*, $(x_1, ..., x_n)$, are *optimal* (maximizing profit) ; this factor price set is given by :

$$\frac{w_i}{P\,Y} = \psi_i\,(x_1, .., x_m) \equiv \frac{\frac{\partial g}{\partial x_i}(x_1, .., x_m)}{\frac{\partial g}{\partial x_1}\cdot x_1 + \frac{\partial g}{\partial x_2}\cdot x_2 + .. + \frac{\partial g}{\partial x_m}\cdot x_m}\ ;\ w_i = P\,\frac{\partial g}{\partial x_i}(x_1, .., x_m),\ i = 1, 2, ., m \quad (91)$$

$\mathbf{w_i}$ - *factor price functions*, RHS, (91), are the money *value marginal* factor productivity equations - or the **Inverse** *factor demand* functions [for competitive general equilibrium in both product (*fixed P*) and factor markets (*fixed* supply of other *factors*, $\mathbf{x_j}, j \neq i$)]

9.1.3. *Existence of Age annual wage functions.* Given a regular (monotone, concave, smooth, homogeneous of degree one) **Labor Aggregator** function, $L_A = f\,(L_1, L_2, ..., L_M)$, giving *Total wage income* (Labor earnings), $W \equiv W_A L_A = w_1 L_1 +, ..., + w_M L_M$, there is *only one set* of *annual wages*, w_i, $i = 1, 2, .., n$, for which Labor *supplies* of *Age* groups, $(L_1, L_2, .., L_M)$, are used *efficiently* (maximizing Total wages); this wage set is given by:

$$\frac{w_i}{W_A L_A} = \Psi_i\,(L_1, .., L_M) \equiv \frac{\frac{\partial f}{\partial L_i}(L_1, .., L_M)}{\frac{\partial f}{\partial L_1}\cdot L_1 + \frac{\partial f}{\partial L_2}\cdot L_2 + .. + \frac{\partial f}{\partial L_M}\cdot L_M}\ ;\ w_i = W_A\,\frac{\partial f}{\partial L_i}(L_1, .., L_M),\ i = 1, 2, ., M \quad (92)$$

$$\frac{w_i\,L_i}{W_A L_A} = \Psi_i\,(L_1, .., L_M)\cdot L_i \equiv \frac{\frac{\partial f}{\partial L_i}(L_1, .., L_M)\cdot L_i}{\frac{\partial f}{\partial L_1}\cdot L_1 + \frac{\partial f}{\partial L_2}\cdot L_2 + .. + \frac{\partial f}{\partial L_M}\cdot L_M} = \varepsilon_i\,,\quad i = 1, 2, ., M\ ;\ \sum_{i=1}^{M}\varepsilon_i = 1 \quad (93)$$

$\mathbf{w_i}$ - *annual wage functions*, RHS, (92), are the money *value marginal* Labor *contributions* of $\mathbf{L_i}$ to Wage sum, W - or **Inverse** *Labor demand* functions [for competitive equilibrium of the Aggregate Labor market (*fixed* W_A) and *fixed* Labor supply of other *ages*, $\mathbf{L_j}, j \neq i$]. RHS, (93), shows that wage functions w_i (92) meet Total Wage (W) *accounting identity.*

CRESH implementations of (92-93) are seen as **Age-wage profiles**, $\mathbf{w_i}$, i=1,..,M, in (25-27), (84), $\mathbf{w_{iJ}}$, (30), (33), and to **partial wage elasticities** in (85-89). Using RHS (91) gives Macro **wages**, $\mathbf{W_{AJ}}$, J=I,II, (37-38), by **CRESH production** function (34-36).

[30]Hotelling calls : $P_i = \varphi_i\,(q_1, ..., q_n)\,C$, $i = 1, .., n$, *demand functions* ; see also Hotelling (1932, p.590).

10 App.C : Canonical Wage Structure Model - CRESH

Card and Lemieux (2001, pp.709) used two **CES Subaggregators**, $\mathbf{L_{AJ}}$: College Labor, C = I, and High-school Labor, H = II - stated in notation, (9), (30), and $\rho_J = \rho$,

$$L_{AJ} : L_{A\,I} = \left[\sum_{i=1}^{M} \alpha_{i\,I}\, L_{i\,I}^{\rho}\right]^{\frac{1}{\rho}} , \;\; L_{A\,II} = \left[\sum_{i=1}^{M} \alpha_{i\,II}\, L_{i\,II}^{\rho}\right]^{\frac{1}{\rho}} ; \;\; -\infty < \rho \leq 1 , \;\; \sigma = \frac{1}{1-\rho} \quad (94)$$

As in existing literature, **Aggregate output** (Y) comes with **CES** function of $L_{A\,I}$, $L_{A\,II}$:

$$Y = \left[\sum_{J=I}^{II} a_J\,(t)\, L_{AJ}^{\rho_y}\right]^{\frac{1}{\rho_y}} ; \;\; \frac{\partial Y}{\partial L_{i\,I}} = \frac{\partial Y}{\partial L_{A\,I}} \cdot \frac{\partial L_{A\,I}}{\partial L_{i\,I}} \;\; ; \;\; \frac{\partial Y}{\partial L_{i\,II}} = \frac{\partial Y}{\partial L_{A\,II}} \cdot \frac{\partial L_{A\,II}}{\partial L_{i\,II}} \;\; ; \;\; \frac{w_{i\,I}}{w_{i\,II}} = \frac{\frac{\partial Y}{\partial L_{i\,I}}}{\frac{\partial Y}{\partial L_{i\,II}}} \quad (95)$$

The *marginal product* (output) of workers in *age group* (i) - with College (I) or High School education (II) - are seen (by chain rule) in (95). Pareto efficient utilization of different labor qualities (I, II) requires that *relative wages*, $\frac{w_{i\,I}}{w_{i\,II}}$, are equated to relative marginal products, RHS, (95). The partial derivatives of the CES functions in (94) and (95) imply that *relative wages* in *same age* group (i), $\frac{w_{i\,I}}{w_{i\,II}}$, satisfy equation (96)[31], LHS:

$$\begin{aligned} \frac{w_{i\,I}\,(t)}{w_{i\,II}\,(t)} &= \frac{a_I\,(t)}{a_{II}\,(t)} \frac{\alpha_{i\,I}}{\alpha_{i\,II}} \left[\frac{L_{A\,I}\,(t)}{L_{A\,II}\,(t)}\right]^{\rho_y-\rho} \left[\frac{L_{i\,I}\,(t)}{L_{i\,II}\,(t)}\right]^{\rho-1} ; \;\; \rho_y - \rho = -\frac{1}{\sigma_y} + \frac{1}{\sigma} , \;\; \rho - 1 = -\frac{1}{\sigma} \quad (96) \\ &\equiv \frac{a_I\,(t)}{a_{II}\,(t)} \frac{\alpha_{i\,I}}{\alpha_{i\,II}} \left[\frac{L_{A\,I}\,(t)}{L_{A\,II}\,(t)}\right]^{-\frac{1}{\sigma_y}} \left[\frac{\lambda_{i\,I}\,(t)}{\lambda_{i\,II}\,(t)}\right]^{-\frac{1}{\sigma}} , \;\; \lambda_{i\,I} = \frac{L_{i\,I}}{L_{A\,I}} , \; \lambda_{i\,II} = \frac{L_{i\,II}}{L_{A\,II}} , \;\; \sum_{i=1}^{M} \lambda_{i\,J} = 1 \quad (97) \end{aligned}$$

which is equivalent to expression (97) - with *Employment* (Supply)[32] *ratios* (Labor proportion of College/High school workers), and using *Age* composition (distribution) *within* College/High school workers ($\lambda_{i\,J}$), (97). Evidently from (97), *larger substitution elasticities* ($\sigma_{\mathbf{y}}$ and σ) imply *smaller* changes in the relative wages, $\frac{\mathbf{w_{i\,I}}}{\mathbf{w_{i\,II}}}$ [or log changes $r_i\,(t)$, (98)], coming from variation in Aggregate *Supply ratios* and Age compositions ($\lambda_{\mathbf{i\,J}}$).

$$r_i\,(t) \equiv log\; \frac{w_{i\,I}\,(t)}{w_{i\,II}\,(t)} \;\; , \quad i = 1 = 26-30, 31-35, .., M = 7 = 56-60 \, ; \;\; I,\, II \qquad (98)$$

[31]Card and Lemieux (2001, p.710, equation 7) presents (96), LHS, in logarithmic form, which is more convenient for parameter estimation purposes. We do not enter estimation - will only discuss the results.

[32]In relative wages (97), the aggregate supply *ratio* (relative supplies) is also seen in *share* form, $\lambda_{A\,J}$:

$$\frac{w_{i\,I}\,(t)}{w_{i\,II}\,(t)} = \frac{a_I(t)}{a_{II}(t)} \frac{\alpha_{i\,I}}{\alpha_{i\,II}} \left[\frac{\lambda_{A\,I}\,(t)}{\lambda_{A\,II}\,(t)}\right]^{-\frac{1}{\sigma_y}} \left[\frac{\lambda_{i\,I}\,(t)}{\lambda_{i\,II}\,(t)}\right]^{-\frac{1}{\sigma}} , \, \lambda_{A\,I} = \frac{L_{A\,I}}{L_{A\,I}+L_{A\,II}} , \, \lambda_{A\,II} = \frac{L_{A\,II}}{L_{A\,I}+L_{A\,II}} , \sum_{J=I}^{II} \lambda_{A\,J} = 1$$

For changes in log shares - $\frac{d\,\lambda_{A\,J}}{\lambda_{A\,J}} \cdot 100$ - of aggregate labor input groups and their relative wages changes, see Katz and Murphy (1992, p.39-40,49,67-68) - where the aggregate labor supplies, $L_{A\,J}\,(t)$, are measured in so-called *'efficiency units'*. No *Age-specific* full-time equivalents $L_{i\,J}$ appear in Katz & Murphy (1992).

For year (t), $r_i\,(t)$ is called a College-High school *premium* or *wage gap* for *age group* (i). Ratios $r_i\,(t)$, $i = 1, ., 7$ is an *Age profile* (98) of *premiums*/wage gaps for *calendar* year (t). We will briefly discuss the *parameters*, (96-97), of the *Age profiles* (98) that Card and Lemieux (2001, p.715, 718) has *calculated* in *years*, 1959,1970,1975,1980,1985,1990,1995, for the United States, and roughly *same* calendar years for United Kingdom and Canada.

For all tree countries, CES parameter estimates, Card & Lemieux (2001, pp.725-27), of ρ, (94), (96), were in the **range** : $\rho = 0.77$ to $\rho = 0.83$ [33], i.e., $\sigma = 4.34$ to $\sigma = 5.88$.

The *estimated* sizes of the *age* specific *efficiency* (intensity) parameters - $\alpha_{i\,I}$ and $\alpha_{i\,II}$ - in (94), (96-97), for the seven (98) age groups, i= 1,.,M=7, are *not* available [34] (reported); they give the two *age-wage profiles* of, $w_{i\,I}\,(t)$, $w_{i\,II}\,(t)$, i= 1, 2,,M, by (94) - as seen below.

In contrast to $\alpha_{i\,I}$, $\alpha_{i\,II}$, the relative *efficiency* (intensity) parameters, a_I, a_{II}, (95-97), are *not* time-invariant for the aggregates $L_{A\,I}$ and $L_{A\,II}$ in the *Production* function (95). Card & Lemieux (p.725) give for the tree countries *estimates* of *year effects*, reflecting changes (technology *shocks*) to the ratios, $\frac{a_I\,(t)}{a_{II}\,(t)}$, in the calendar years above. These year effects are rising, and next replaced by *linear trends* : $\frac{a_I\,(t)}{a_{II}\,(t)} = \beta\, t$, $\beta \in (0.017, 0.020)$ for US, $\beta \in (0.021, 0.018)$ for UK, $\beta \approx 0$ for Canada. These trend *estimates* (β) are combined with the final estimation of $\rho_{\mathbf{y}}$, (95). Card & Lemieux (p.727) present *two* estimates of $\rho_{\mathbf{y}}$ [depending on sizes of the Aggregate Supply indexes for College Labor and High-school Labor, 1. Katz-Murphy indexes, 2. L_{AJ} in (94)]. Thus we see the *estimates* for the US : 1. $\rho_y = 0.59$, 2. $\rho_y = 0.52$, i.e., $\sigma_y = 2.44$ or $\sigma_y = 2.08$. For UK: 1. $\rho_y = 0.53$, 2. $\rho_y = 0.66$, i.e., $\sigma_y = 2.13$ or $\sigma_y = 2.94$. For Canada: 1. $\rho_y = 0.93$, 2. $\rho_y = 0.87$, i.e., $\sigma_y = 14.29$ or $\sigma_y = 7.69$ (Canadian ρ_y are imprecise estimates, also $\rho_y = 0.82$, $\sigma_y = 3.57$).

By using second estimate (2) of (ρ_y , σ_y), second trend coefficient of (β), common estimate of ($\rho_I = \rho_{II} = \rho$, σ), together with *relative Aggregate* Labor Supplies, $L_{A\,I}\,(t)/L_{A\,II}\,(t)$, and *relative Age-group* Supplies, $L_{i\,I}\,(t)/L_{i\,II}\,(t)$, [*relative Age* distributions, $\lambda_{i\,I}\,(t)/\lambda_{i\,II}(t)$] we obtain with (96-97) their *Age-relative wage* profile, $\mathbf{w_{i\,I}\,(t)/w_{i\,II}\,(t)}$, $i = 1, ., M = 7$, for

[33]In **Table 4** (Col. 3), the range of the CRESH age-specific ρ_i was : $\rho_i = 0.5$ $\rho_i = 0.8$; but $\rho_i = 0.5$ applied only to two *age* groups, 40-44, 45-49. CES parameter $\rho = 0.8$ ($\sigma = 5$) can fit any data of relative wages pretty well ; cf. Guest & Jensen (2016, p.31, Fig.4 - misprint p.32: interchange titles of Fig 5,6).

[34]Card & Lemieux (p.713, eq.12a-12b) show how, $\alpha_{i\,I}$, $\alpha_{i\,II}$, ($\alpha'_j s, \beta'_j s$), ρ (η), are estimated - and used to construct estimates of Aggregate Labor supplies, $L_{A\,I}\,(t)$, $L_{A\,II}\,(t)$, (C_t, H_t), in (94) ; cf. footnote 26.

the calendar *year* (t) - or in log version, *Age-relative wage* profile (98), $\mathbf{r_i}\,(\mathbf{t})$, $i = 1, ., M = 7$, called the (*College premiums* - wage *gaps*) for calendar *year* (t).

The *shape* of *Age-relative wage* profiles, $\frac{\mathbf{w_{i\,I}\,(t)}}{\mathbf{w_{i\,II}\,(t)}}$, [or $\mathbf{r_i}\,(\mathbf{t})$], $i = 1, , M$, have changed over years 1970,1975,1980,1985,1990,1995 - but alike for three countries, Card & Lemieux (p.718). It has shifted upwards before 1980 - and *rotated* (twisted) after 1980-85, with *younger* workers (31-35, 36-40) *rising* much more than the older workers (46-50, 51-55).

Apart from 'biased' *technology* trends, $a_I\,(t)/a_{II}\,(t) = \beta\, t$, Card & Lemieux (p.707) see a *deceleration* (slower increases) in *relative* College Labor *supplies* since 1980 as the driving force behind the increased[35] *relative* wages (96-98). But behind such relative wages,[36] (96-98), we also want economic-analytically to know exactly what occur - in a consistent way - to the corresponding *calendar* year (t) *Age-wage profiles* : $w_{i\,I}\,(t)$, i= 1, 2,..,M, and, $w_{i\,II}\,(t)$, i= 1, 2,..,M. We will match CES version of $w_{iJ}\,(t)$, (30), to (94-97).

[35]See Acemoglu & Autor (2011, p.1052); Goldin & Margo (1992, p.7), Murphy & Welch (1992, p.294).

[36]As background for CRESH production function, (34-36), the general CES version is discussed here. The CES production functions are (normalizing, $\sum_{J=I}^{VI} a_J = 1$, may need γ for dimensional reasons) :

$$Y \; = \; \gamma \left[\sum_{J=I}^{VI} a_J\,(b_J X_J\,)^{\,\rho} \right]^{\frac{1}{\rho}} \; ; \; \gamma > 0\,; \quad \sum_{J=I}^{VI} a_J \; = 1\,; \;\; -\infty < \rho \leq 1\,, \;\; \sigma = \tfrac{1}{1-\rho}$$

where b_J are *factor-augmenting* technology terms - parameters (constants) or specified functions of time (trends). The expressions, $\mathbf{b_J X_J}$, are referred to as the *factor supplies* shown (measured) in "*efficiency units*". Usually, the variables (quantities), $X_J, J = I, .., VI$, have their own *units* of *measurements* (e.g., labor, capital, etc.), which are entirely different matters (and problems) than attaching *factor-augmenting* terms (parameters, functions, a_J) to each variable X_J in discrete or continuous time. For our purposes, we did neither consider any factor augmenting terms (b_J) involved in $L_{A\,J}$, (95), nor (b_{iJ}) in $L_{i\,J}$, (94).

Autor et al. (1998, p.1176-79) use such CES two-factor labor augmenting (b_I, b_{II}) version of (95), giving the relative wages (ratios of marginal products of two labor types) as follows [notation, (96-97)]:

$$\frac{w_I\,(t)}{w_{II}\,(t)} = \frac{a_I\,(t)}{1 - a_I\,(t)} \left[\frac{b_I\,(t)}{b_{II}\,(t)}\right]^{\rho} \left[\frac{L_I\,(t)}{L_{II}\,(t)}\right]^{-\frac{1}{\sigma}} \;\; \equiv \; d(t) \left[\frac{L_I\,(t)}{L_{II}\,(t)}\right]^{-\frac{1}{\sigma}} \quad ; \; \rho = \frac{\sigma - 1}{\sigma} \;\; , \;\; \sigma = \sigma_y \;, \; \rho = \rho_y$$

$w_I = \frac{\partial Y}{\partial L_I} = a_I(t)\, b_I^{\,\rho}\,(t) \left[\frac{Y}{L_I}\right]^{\rho-1}$, $w_{II} = \frac{\partial Y}{\partial L_{II}} = a_{II}(t)\, b_{II}^{\,\rho}\,(t) \left[\frac{Y}{L_{II}}\right]^{\rho-1}$. Such technology term $\left[\frac{b_I\,(t)}{b_{II}\,(t)}\right]^{\rho}$ may be included in (101). No *Age-specific* full-time equivalents, $L_{i\,J}$ appear in Autor et al. (1998)

The 'parameter elements' within the composite variable, $d(t)$, are used to reflect technological- and relative factor demand shifts that may favor college equivalents, $L_I\,(t)$, raising the college premium/wage gap : $r\,(t) \;\equiv\; log\; \frac{w_I\,(t)}{w_{II}\,(t)}$. Using Katz & Murphy (1992, p.69) point estimate of $\sigma_y = 1/0.709 = 1.41$, and lower/upper limits of σ_y (1, 2), Autor et al. (1998), calculate - for σ_y =1, 1.4, 2 - and with data, $\frac{L_I\,(t)}{L_{II}\,(t)}$, the *college premium*, $r\,(t)$, and implied relative demand *shifts*, $d(t)$, for decades in period 1940-1996.

In (37-38), the *exogenous* $\mathbf{W_{A\,J}}$ referred to *Average wage* of particular Subaggregates of workers, $L_{A\,J}$. With a single-sector (output), aggregate **CES** production function, (95), the efficiency in production implies that $\mathbf{W_{A\,J}}$ are *marginal value products* of **Labor**, i.e.,

$$\frac{W_{A\,I}}{P} = \frac{\partial Y}{\partial L_{A\,I}} = a_I(t)\left[\frac{Y}{L_{A\,I}}\right]^{\rho_y-1} \; ; \; \frac{W_{A\,II}}{P} = \frac{\partial Y}{\partial L_{A\,II}} = a_{II}(t)\left[\frac{Y}{L_{A\,II}}\right]^{\rho_y-1} \; ; \; Y = \left[\sum_{J=I}^{II} a_J(t)\, L_{AJ}^{\rho_y}\right]^{\frac{1}{\rho_y}} \tag{99}$$

Thus with (99) and CES (94), **calendar** *wages* of *Age* group (i), $w_{iJ}\,(t)$, (30), becomes:

$$w_{i\,I}\,(t) = P\cdot a_I(t)\left[\frac{Y}{L_{A\,I}}\right]^{\rho_y-1}\alpha_{i\,I}\,\lambda_{i\,I}^{\rho-1}(t) \;\; ; \;\; w_{i\,II}\,(t) = P\cdot a_{II}(t)\left[\frac{Y}{L_{A\,II}}\right]^{\rho_y-1}\alpha_{i\,II}\,\lambda_{i\,II}^{\rho-1}(t) \;\; ; \;\; i=1,..,M \tag{100}$$

Accordingly, (100) is consistent with *relative* wages, LHS (101) - formally similar to (97):

$$\frac{w_{i\,I}\,(t)}{w_{i\,II}\,(t)} = \frac{a_I(t)}{a_{II}(t)}\frac{\alpha_{i\,I}}{\alpha_{i\,II}}\left[\frac{L_{A\,I}\,(t)}{L_{A\,II}\,(t)}\right]^{-\frac{1}{\sigma_y}}\left[\frac{\lambda_{i\,I}\,(t)}{\lambda_{i\,II}\,(t)}\right]^{-\frac{1}{\sigma}} ,\; L_{A\,I} = L_{\,I} = \sum_{i=1}^{M} L_{i\,I} \; ; \; L_{A\,II} = L_{\,II} = \sum_{i=1}^{M} L_{i\,II} \tag{101}$$

Note. In (99-101), *Aggregate* Labor *supplies* ($L_{A\,I}$, $L_{A\,II}$) are Age-group *sums* (RHS (101).

We now need to scrutinize the concepts and the actual numbers of the *Aggregate* Labor *supply* variables, $L_{A\,J}$, $J{=}I,II$, that appear in the CES functions, (94-95). The CES *Subaggregator* formulas (94) are often called Aggregate *Supply indexes* of College/High school Labor resources. Clearly, larger values of CES parameter ρ (σ) affect the isoquant maps of (94) analogously to the role of ρ_y (σ_y) for the isoquant maps of (95), implying e.g., that *larger* supply index numbers $L_{A\,I}$ are attained with *smaller* sizes of L_{iI} in (94); but such larger abstract-theoretical $L_{A\,I}$ total supply numbers are neither directly observable nor satisfy simple labor accounting identities stated in RHS (101). With well-defined *measuring units* as 'full-time worker (College/High school) equivalents', the direct sum (accounting) of sub-groups in RHS (101) are important to satisfy - as in **Table 2** (Col.5). In short, Subaggregator formulas (94) are *not* used to obtain (predict) *Total quantities*, $L_{A\,J}$, but (94) are used to *generate* sub-group *wages* to form Total Wage Income, (25-27).

Only *derivatives* (31-32) of *Aggregator* functions, $L_{AJ} = f_J$, are used in *imputing wages* w_{iJ}, (100), to the *age-groups* (i) of Total Labor supply,[37] $L_{AJ} = \sum_{i=1}^{M} L_{i\,J}$, RHS (101).

The *size* (level) of the Average wages - $\mathbf{W_{AJ}}$ were not explained *economically.* The CES Production functions (95) attempted such economic explanation of, W_{AJ}, through the *marginal products* of Labor, (99), L_{AJ}, $J = I, II$, which implied the calendar *Age-wage* profiles, $w_{i\,J}\,(t)$, $i = 1, ., M$, (100), of College/High school Labor, $J = I, II$. *Shifting* (twisting, *rotating*) of the *two* calendar CES *Age-wage* profiles is seen by (100) to follow from: 1.*divergent* evolutions of the (sum) *Totals*,[38] $L_{A\,I}\,(t)$, $L_{A\,II}\,(t)$, 2.*divergent* evolutions of the **Age distributions**, λ_{iJ}, $i = 1, ., M$, *within* the two categories, $J = I, II$.

With CRESH, CES *calendar* year *Age-wage* profiles (100), are replaced by inserting - *marginal value products* of Labor, (99) - into CRESH **Age-wage profile** formula (33):

$$w_{i\,I}\,(t) = P \cdot a_I(t) \left[\frac{Y}{L_{A\,I}}\right]^{\rho_y - 1} \frac{\alpha_{i\,I}\,\rho_{iI}\,\lambda_{iI}^{\rho_{iI}-1}}{\sum_{i=1}^{M} \alpha_{i\,I}\,\rho_{iI}\,\lambda_{iI}^{\rho_{iI}}} \;\; ; \;\; w_{i\,II}\,(t) = P \cdot a_{II}(t) \left[\frac{Y}{L_{A\,II}}\right]^{\rho_y - 1} \frac{\alpha_{i\,II}\,\rho_{iII}\,\lambda_{iII}^{\rho_{iII}-1}}{\sum_{i=1}^{M} \alpha_{i\,II}\,\rho_{iII}\,\lambda_{iII}^{\rho_{iII}}} \tag{102}$$

Evidently, more elaborate **shifting**[39] by two CRESH **calendar** year **Age wage** profiles.[40] Replacing CES, (99) by (37-38) give in (102) Age wage profiles affected by (K_{III}, K_{IV}, K_V).

[37]Although relative wages in LHS (101) formally look 'similar' to (97), there is an ambiguity about the numerical size of, $L_{A\,I}$, $L_{A\,II}$, appearing in Sub-aggregators, (94), Production function (95), and relative wages, (96-97). As discussed above, the absolute size of, $L_{A\,I}$, $L_{A\,II}$, are irrelevant for using Sub-aggregators, (94) to age-wage imputations, where only **age distributions**, λ_{iJ}, i=1,..,M, mattered cf. (31-32). However, the total imputed wage sums are: $W_{AJ}\,L_{AJ} = \sum_{i=1}^{M} w_{iJ} L_{iJ} \equiv w_J L_J \equiv W_J$, $J = I, II$, i.e., L_{AJ} as in RHS (101) - and not as L_{AJ} determined in (94). Thus L_{AJ} in (95) is neither (94), but RHS (101) - no ages are involved in the factor substitutions by (95). The text above (p.42) mentioned two estimates of $(\rho_y\,,\,\sigma_y)$ by : 1. Katz-Murphy indexes, 2. L_{AJ} in (94). Thus, $(\rho_y\,,\,\sigma_y)$ estimates by the latter Labor supply numbers (94) are inadequate - instead we wanted estimates of $\alpha_{i\,J}$, cf. footnote 23.

[38]The time path of the ratio, $\frac{L_{A\,I}\,(t)}{L_{A\,II}\,(t)}$ is called "*intercohort shifts* in the relative supply of highly educated workers", "intercohort trend in educational attainment", and $\lambda_{iI}(t), \lambda_{iII}(t)$, $i = 1, ., M$, are called "differences in *age distributions* of educational attainment", Card & Lemieux (p.707). Cohorts means *calendar* year (t) total *Labor supplies*, $L_{A\,J}\,(t)$, $J = I, II$. Cohorts in the sense of Labor *supplies*, $L_i^*(T)$, [life-cycle ages (i)], and life time supplies, $L^*(T)$, cf. (44), are not seen in Card & Lemieux (2001).

[39]Analysis and explanations of the 'relative demand shifts'/'trends' behind college premium/wage inequality (note 25) need of a variety of models, including trade, cf. Borjas & Ramey (1994, p.12), Borjas et al. (1997, p.11, 40-41), Topel (1997, p.68).

[40]A *pure* labor economy model by the production and substitution with only *two Labor* factors (categories), $L_{A\,I}\,(t), L_{A\,II}\,(t)$, (95), is of course an abstraction for explaining, W_{AJ}, (99).

References

Acemoglu, D. and Autor, D. (2011), *Skills, Tasks and Technologies: Implications for Employment and Earnings*, pp.1043-1171 in : Ashenfelter, O.C. and Card, D., eds., (2011), Vol.4B.

Allen, R. G. D. (1938), *Mathematical Analysis for Economists*, London, MacMillan.

Arrow, K.J., Chenery, H.B., Minhas, B.S. and Solow, R.M. (1961), Capital-Labor Substitution and Economic Efficiency, *Review of Economics and Statistics*, **43**, 225-50.

Ashenfelter, O.C. and Card, D., eds., (2011), *Handbook of Labor Economics*, Vol.4A-4B, Amsterdam , Elsevier.

Ashenfelter, O.C. and Card, D., eds., (1999), *Handbook of Labor Economics*, Vol. 3A-3C, Amsterdam , Elsevier.

Autor, D.H., Katz, L.F. and Kruger, A.B. (1998), Have Computers Changed the Labor Market, *Quarterly Journal of Economics*, **113**, 1169-1213.

Barth, E., Davis, J. and Freeman, R.B. (2018), Augmenting the Human Capital Earnings Equation with Measures of Where People Work, *Journal of Labor Economics,* **36**, S71-97.

Berger, M. C. (1984), Cohort Size and the Earnings Growth of Young Workers, *Industrial and Labor Relations Review*, **37**, 582-591.

Berger, M. C. (1985), The Effect of Cohort Size on Earnings Growth : A Re-examination of the Evidence, *Journal of Political Economy,* **93**, 561-573.

Berndt, E. R. and Christensen, L. R. (1973), The Internal Structure of Functional Relationships: Separability, Substitution, and Aggregation, *Review of Economic Studies*, **40**, 403-410.

Berndt, E. R. and Christensen, L. R. (1974), Testing the Existence of a Consistent Aggregate Index of Labor Inputs, *American Economic Review*, **64**, 391-404.

Blaug, M. (1967), The Private and the Social Returns on Investment in Education : some Results of Great Britain, *Journal of Human Resources,* **2**, 330-346.

Bowles, S. (1970), Aggregation of Labor Inputs in the Economics of Growth and Planning: Experiments with a Two-Level CES Function, *Journal of Political Economy,* **78**, 68-81.

Borjas, G.J. (2020), *Labor Economics*, 8. Ed. , McGraw-Hill, New York.

Borjas, G.J. , ed. (2014), *The Economics of Labor*, Vol. I-IV, Routledge.

Borjas, G.J., Grogger, J. and Hansen, G.H. (2012), On Estimating Elasticities of Substitution, *Journal of European Economic Association*, **10**, 198-210.

Borjas, G.J. (2003), The Labor Demand Curve is Downward Sloping: Reexamining the Impact of Immigration on the Labor Market, *Quarterly Journal of Economics*, **118**, 1335-1374.

Borjas, G.J., Freeman, R.B. and Katz, L.F. (1997), How Much Do Immigration and Trade Affect Labor Outcomes ?, *Brookings Papers on Economic Activity*, 1997 No.1, 1-90.

Borjas, G.J. and Ramey, V.R. (1994), Time-Series Evidence on the Sources of Trends in Wage Inequality, *American Economic Review*, **84**, No.2, 10-16.

Borjas, G.J. (1986), The Sensitivity of Labor Demand Functions to Choice of Dependent Variable, *Review of Economics and statistics,* **68**, 58-66.

Card, D. and Lemieux, T. (2001), Can Falling Supply Explain the Rising Return to College for Younger Men? A Cohort-Based Analysis, *Quarterly Journal of Economics,* **116**, 705-746.

Caselli, F. and Feyrer, J. (2007), The Marginal Product of Capital, *Quarterly Journal of Economics*, **122**, 535-568.

Chiswick, C. U. (1985), The Elasticity of Substitution Revisited: The Effects of Secular Changes in Labor Force Structure, *Journal of Labor Economics*, **3**, 490-507.

Conlon, J.R. (1993), Separability and Separability Flexibility of the CRESH Function, *The Journal of Productivity Analysis*, **4**, 371-377.

Cornes, R. (1992), *Duality and Modern Economics*, Cambridge Univ. Press, Cambridge.

Creedy, J. and Guest, R., Eds., (2007), *New Developments in the Economics of Ageing*, Edward Elgar.

Cutler, D.C., Poterba, J.M., Sheiner, L.M. and Summers, L.H. (1990), An Ageing Society: Opportunity or Challenge ?, *Brookings Papers on Economic Activity*, 1990 No.1, 1-73.

Diewert, W.E. (1974), Applications of Duality Theory, 106-171, in: Intriligator, M.D. and Kendrick, D.A., (Eds), *Frontiers of Quantitative Economics*, Vol. II, North-Holland, Amsterdam.

Dougherty, R.S. (1972), Estimates of Labor Aggregation Functions, *Journal of Political Economy*, **80**, 1101-1119.

Easterlin, R. (1978), What will 1984 be like ? Socio-economic implications of recent twists in age structure, *Demography*, **15**, 397-432.

Easterlin, R.A., Macdonald, C. and Macunovich, D.J. (1990), How Have American Baby Boomers Fared : Earnings and Economic Well-Being of Young Adults, 1964-1987, *Journal of Population Economics*, **3**, 277-290.

Edin, P.A. and Holmlund, B. (1995), *The Swedish Wage Structure : The Rise and Fall of Solidarity Wage Policy ?*, 307-343, in : Freeman, R. and Katz, L.F., eds., (1995).

Freeman, R.B. and Katz, L.F., eds., (1995), *Differences and Changes in Wage Structures*, University of Chicago Press, Chicago, IL.

Freeman, R.B. (1979), The Effect of Demographic Factors on Age-Earnings Profiles, *Journal of Human Resources,* **14**, 289-318.

Freeman, R.B. (1977), The Decline in the Economic Rewards to College Education, *Review of Economics and statistics,* **59**, 18-29.

Freeman, R.B. (1976), *The Over-Educated American*, Academic Press, New York.

Freeman, R.B. (1971), *The Market for College Trained Manpower*, Harvard University Press, Cambridge, Mass.

Goldin, C. and Margo, R.A. (1992), The Great Compression: The Wage Structure in the United States at Mid-Century, *Quarterly Journal of Economics*, **107**, 1-34.

Green, H.A.J. (1964), *Aggregation in Economic Analysis: An Introductory Survey*, Princeton University Press, Princeton.

Guest, R. and Parr, N. (2020), Fertility, immigration, and lifetime wages under imperfect labor substitution, *Journal of Demographic Economics*, **86**, 503-532.

Guest, R. (2007), Can OECD Countries Afford Demographic Change ? , *Australian Economic Review*, **40** (2), 1-16.

Guest, R. and Shacklock, K. (2005), The Impending Shift to an Older Mix of Workers: Perspectives from the Management and Economics Literatures, *International Journal of Organisational Behaviour*, **10** (3), 713-728.

Guest, R. and Jensen, B.S. (2016), Applying a CRESH Aggregate Labour Index to Generate Age-Wage Profiles, *Applied Economic Letters*, **23** (1), 27-33.

Hamermesh, D. S. 1993, *Labor Demand*. Princeton: Princeton University Press.

Hanoch, G. (1978), Polar Functions with Constant Two Factors-One Price Elasticities, 287-309, in: Fuss, M. and McFadden, D., (Eds), *Production Economics: A Dual Approach to Theory and Applications*, Vol.1, North-Holland, Amsterdam.

Hanoch, G. (1971), CRESH Production Functions, *Econometrica*, **39** (5), 695-712.

Hanoch, G. (1967), An Economic Analysis of Earnings and Schooling, *Journal of Human Resources,* **2**, 310-329.

Hicks, J. (1970), Elasticities of Substitution Again: Substitutes and Complements, *Oxford Economic Papers*, **22**, 289-296.

Hu, S.C., Chen, K.M. and Chen, L.T., T. (2000), Demographic Transition and Social Security in Taiwan, *Population and Development Review*, **26**, Supplement, 117-138.

ILO (2013), *Key Indicators of the Labor Market, 2012 Edition*, International Labor Organisation, Geneva, Switzerland.

Jensen, B.S., Lehmijoki, U. and Rovenskya, E. (2015), Non-Homothetic Multisector Growth Models,, *Review of Development Economics*, **19**, 221-243.

Jensen, B.S., and Lehmijoki, U. (2011), Homothetic Multisector Growth Models, 289–318, in: *Frontiers of Economics and Globalization*, **11**, La Grandville, O. de, (Ed.), Emerald, United Kingdom.

Jensen, B.S., de Boer, P., van Daal, J. and Jensen, P.S. (2011), Global restrictions on the parameters of the CDES indirect utility function, *Journal of Economics*, **102**, 217-235.

Jensen, B.S., and Larsen, M.E. (2005), General Equilibrium Dynamics of Multi-Sector Growth Models, *Journal of Economics*, Suppl. **S. 10**, 17-56.

Jensen, B.S. (2003), Walrasian General Equilibrium Allocations and Dynamics in Two-Sector Growth Models, *German Economic Review*, **4** , 53-87.

Jensen, B.S. (1994), *The Dynamic Systems of Basic Economic Growth Models*, Kluwer Academic Publishers, Dordrecht/Boston.

Katz, L.F. and Autor, D.H. (1999), *Changes in the Wage Structure and Earnings Inequality*, 1463-1555, in : Ashenfelter, O.C. and Card, D., Ed., (1999), Vol. 3A.

Katz, L.F. and Murphy, K.M. (1992), Changes in Relative Wages, 1963-1987 : Supply and Demand Factors, *Quarterly Journal of Economics*, **107**, 35-78.

Kitao, S. (2015), Fiscal Cost of Demographic Transition in Japan, *Journal of Economic Dynamics and Control*, **54**, 37-58.

Lam, D. (1989), Population Growth, Age Structure, and Age-Specific Productivity : Does a Uniform Age Distribution Minimize Lifetime Wages ? *Journal of Population Economics*, **2**, 189-210.

Lucas, R.E., (1988), On the Mechanics of Economic Development, *Journal Monetary Economics,* **22**, 3-42.

McFadden, D., (1978), Cost, Revenue, and Profit Functions, 1-109, in: Fuss, M. and McFadden, D., (Eds), *Production Economics: A Dual Approach to Theory and Applications*, Vol.1, North-Holland, Amsterdam.

Mokyr, J. (1990), *The Lever of Riches: Technological Creativity and Economic Progress*, Oxford University Press, New York.

Murphy, K.M. and Welch, F. (1992), The Structure of Wages, *Quarterly Journal of Economics*, **107**, 285-326.

Prskawetz, A., T. Fent and R. Guest (2008), Workforce Ageing and Labor Productivity. The role of supply and demand for labor in the G7, *Population and Development Review*, **34** (Supplement), 298-323.

Rojas, J. (2005), Life-cycle earnings, cohort size effects and social security: a quantitative exploration, *Journal of Public Economics*, **89**, 465-485.

Romer, D. (2019), *Advanced Macroeconomics*, 5. Ed., McGraw-Hill, New York.

Rosen, S. (1972), Learning and Experience in the Labor Market, *Journal of Human Resources*, **7**, 326-342.

Rosen, S. (1976), A Theory of Life Earnings, *Journal of Political Economy,* **84**, S45-S66.

Sato, K. (1967), A Two-Level Constant-Elasticity-of-Substitution Production Function, *Review of Economic Studies*, **34**, 201-218.

Sato, R. and Koizumi, T. (1970), Substitutability, Complementarity and the Theory of Derived Demand, *Review of Economic Studies*, **37**, 107-118.

Sato, R. and Koizumi, T. (1973), On the Elasticities of Substitution and Complementarity, *Oxford Economic Papers*, **25**, 44-56.

Sato, R. and Koizumi, T. (1973), The Production Function and the Theory of Distributive Shares, *American Economic Review*, **63**, 484-489.

Smith, A. (1790, 1961), *An Inquiry Into The Nature And Causes Of The Wealth of Nations*, 5. Ed. (ed. E. Cannan). Vol. 1-2, Methuen, London.

Solow, R. M. (2000), *Growth Theory*, Oxford University Press, Oxford.

Tinbergen, J. (1974), Substitution of Graduate by Other Labor, *Kyklos*, **27**, 217-226.

Topel, R.H. (1997), Factor Proportions and Relative Wages: The Supply-Side Determinants of Wage Inequality, *The Journal of Economic Perspectives*, **11**, 55-74.

United Nations (2015) *World Population Prospects, 2015 Revision*, United Nations, N.Y.

Uzawa, H. (1962), Production Functions with Constant Elasticities of Substitution, *Review of Economic Studies*, **29**, 291-299.

Van Daal, J. and Merkies, A.H.Q.M. (1984), *Aggregation in Economic Analysis : From Individual to Macro Relations*, D. Reidel (Kluwer) Publ., Dordrecht/Boston.

Weiss, R. D. (1977), Elasticities of Substitution Among Capital and Occupations in U.S. Manufacturing, *Journal of the American Statistical Association*, **72**, 746-771.

Welch, F. (1979), Effects of Cohort Size on Earnings : The Baby Boom Babies Financial Bust, *Journal of Political Economy,* **87**, (No.5), S65-S97.

Wold, H. (Juren, L.) (1953), *Demand Analysis*, John Wiley & Sons, New York.

Wold, H. (1943, 1944), A Synthesis of Pure Demand Analysis : Part I-III, *Skandinavisk Aktuarietidsskrift,* **26, 27**, 85-144, 220-275, 69-120.