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ABSTRACT

Structural Change in a Multi-Sector Model of Growth*

We study a multi-sector model of growth with differences in TFP growth rates across sectors and derive sufficient conditions for the coexistence of structural change, characterized by sectoral labor reallocation, and balanced aggregate growth. The conditions are weak restrictions on the utility and production functions commonly applied by macroeconomists. Per capita output grows at the rate of labor-augmenting technological progress in the capital-producing sector and employment moves to low-growth sectors. In the limit all employment converges to two sectors, the slowest-growing consumption-goods sector and the capital-goods sector.

JEL Classification: O41, O14

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1 Introduction

Economic growth takes place at uneven rates across different sectors of the economy. This paper has two objectives related to this fact, (a) to derive the implications of different sectoral rates of total factor productivity growth for structural change, the name given to the shifts in industrial employment shares that take place over long periods of time, and (b) to show that even with ongoing structural change, the aggregate variables can be on a balanced growth path. The restrictions needed to yield structural change consistent with the facts and constant growth are weak restrictions on functional forms that are frequently imposed by macroeconomists in related contexts.

We obtain our results in a baseline model of many consumption goods and a single capital good, supplied by a sector that we label manufacturing and that produces also a consumption good. Our results, however, are consistent with the existence of many capital goods and many intermediate goods under some reasonable restrictions. Production functions in our model are identical in all sectors except for their rates of TFP growth and each sector produces a differentiated good that enters a constant elasticity of substitution (CES) utility function. We show that a low (below one) elasticity of substitution across final goods leads to shifts of employment shares to sectors with low TFP growth. In the limit the employment share used to produce consumption goods vanishes from all sectors except for the slowest-growing one, but the employment shares used to produce capital goods and intermediate goods converge to non-trivial stationary values. If the utility function in addition has unit inter-temporal elasticity of substitution, during structural change the aggregate capital-output ratio is constant and the aggregate economy is on a steady-state growth path.

Our results contrast with the results of Echevarria (1997), Laitner (2000), Caselli and Coleman (2001) and Gollin et al. (2002) who derived structural change in a two- or three-sector economy with non-homothetic preferences. Our results also contrast with the results of Kongsamut et al. (2001) and Foellmi and Zweimuller (2004), who derived simultaneous constant aggregate growth and structural change. Kongsamut et al. obtain their results by imposing a restriction that maps some of the parameters of their Stone-Geary utility function onto the parameters of the production functions, violating one of the most useful conventions of modern macroeconomics, the complete independence of preferences and technologies. Foellmi and Zweimuller (2002) obtain their results by assuming endogenous growth driven by the introduction of new goods into a hierarchic utility function. Our restrictions are quantitative restrictions on a conventional CES utility function that maintains the independence of the parameters of preferences and technologies.

Our results confirm Baumol's (1967) claims about structural change. Baumol divided the economy into two sectors, a "progressive" one that uses new technology and grows at some constant rate and a "stagnant" one that uses labor as the only input and produces services as final output. He then claimed that the production costs and prices of the stagnant sector should rise indefinitely, a process known as "Baumol's cost disease," and labor should move in the direction of the stagnant sector. Baumol controversially also claimed that the economy's growth rate will be on a declining trend, as more weight is shifted to the stagnant sector, a claim that contrasts with our finding that the economy is on a steady-state growth path.[1]

In the more recent empirical literature two competing explanations (which can coexist) have been put forward for structural change. Our explanation, which is sometimes termed "technological" because it attributes structural change to different rates of sectoral TFP growth, and a utility-based explanation, which requires different income elasticities for different goods and can yield structural change even with equal TFP growth in all sectors. Baumol et al. (1985) provide empirical evidence at the 2-digit industry level, consistent with our model, to support Baumol's (1967) claims about employment reallocations between progressive and stagnant sectors. Kravis et al. (1983) also present evidence that favours the technological explanation, at least when the comparison is between manufacturing and services. Two features of their data that are satisfied by the technological explanation proposed in this paper are (a) relative prices reflect differences in TFP growth rates and (b) real consumption shares vary a lot less over time than nominal consumption shares. Our model is also consistent with observed positive correlation between employment growth and relative price inflation across two-digit sectors[2] and with historical OECD evidence presented by Kuznets (1966) and Maddison (1980) for one-digit sectors.

Section 2 describes our model of growth with many sectors and sections 3 and 4 respectively derive the conditions for structural change and the conditions for balanced aggregate growth equilibrium. In sections 5 and 6 we study two extensions of

[1] Ironically, we get our result because we include capital in our analysis, a factor left out by Baumol (1967, p.417) "primarily for ease of exposition ... that is [in]essential to the argument".

[2] These correlations are shown in a longer version of this paper that circulated as working paper. See Ngai and Pissarides (2004).

our benchmark model, one where consumption goods can also be used as intermediate inputs and one where there are many capital goods.

2 An economy with many sectors

The benchmark economy consists of an arbitrary number of m sectors. Sectors $i = 1, ..., m-1$ produce only consumption goods. The last sector, which is denoted by m and labeled manufacturing, produces both a final consumption good and the economy's capital stock. Manufacturing is the numeraire.[3]

We derive the equilibrium as the solution to a social planning problem. The objective function is

$$U = \int_0^\infty e^{-\rho t} v\left(c_1, .., c_m\right) dt, \tag{1}$$

where $\rho > 0$, $c_i \geq 0$ are per-capita consumption levels and the instantaneous utility function $v(.)$ is concave and satisfies the Inada conditions. The constraints of the problem are as follows.

The labor force is exogenous and growing at rate ν and the aggregate capital stock is endogenous and defines the state of the economy. Sectoral allocations are controls that satisfy

$$\textstyle\sum_{i=1}^m n_i = 1; \qquad \sum_{i=1}^m n_i k_i = k, \tag{2}$$

where $n_i \geq 0$ is the employment share and $k_i \geq 0$ is the capital-labor ratio in sector i, and $k \geq 0$ is the aggregate capital-labor ratio. There is free mobility for both factors.

All production in sectors $i = 1, ..., m-1$ is consumed but in sector m production may be either consumed or invested. Therefore:

$$c_i = F^i\left(n_i k_i, n_i\right) \qquad \forall i \neq m \tag{3}$$

$$\dot{k} = F^m(n_m k_m, n_m) - c_m - (\delta + \nu) k \tag{4}$$

where $\delta > 0$ is the depreciation rate, production function $F^i(.,.)$ is constant return to scale, has positive and diminishing returns to inputs, and satisfies Inada conditions.

The social planner chooses the allocation of factors n_i and k_i across m sectors through a set of *static efficiency conditions,*

$$v_i/v_m = F_K^m/F_K^i = F_N^m/F_N^i \qquad \forall i. \tag{5}$$

[3]The label manufacturing is used for convenience. Although in the standard industrial classifications our capital-goods producing sector belongs to manufacturing, some sectors classified as manufacturing in the data (e.g. food and clothing) fall into the consumption category of our model.

The allocation of output to consumption and capital are chosen through a *dynamic efficiency condition,*

$$-\dot{v}_m/v_m = F_K^m - (\delta+\rho+\nu)\,. \tag{6}$$

where F_N^i and F_K^i are the marginal products of labor and capital in sector i.[4] By (5), the rates of return to capital and labor are equal across sectors.

In order to focus on the implications of different rates of TFP growth across sectors we assume production functions are identical in all sectors except for their rates of TFP growth:

$$F^i = A_i F\left(n_i k_i, n_i\right); \quad \dot{A}_i/A_i = \gamma_i; \quad \forall i, \tag{7}$$

With these production functions, we show in the Appendix that static efficiency and the resource constraints (2) imply

$$k_i = k; \qquad p_i = v_i/v_m = A_m/A_i; \qquad \forall i, \tag{8}$$

where p_i is the price of good i in the decentralized economy (in terms of the price of the manufacturing good, $p_m \equiv 1$).

Utility function has constant elasticities both across goods and over time:

$$v\left(c_1,...,c_m\right) = \frac{\phi\left(.\right)^{1-\theta}-1}{1-\theta}; \quad \phi\left(.\right) = \left(\textstyle\sum_{i=1}^m \omega_i c_i^{(\varepsilon-1)/\varepsilon}\right)^{\varepsilon/(\varepsilon-1)} \tag{9}$$

where $\theta, \varepsilon, \omega_i > 0$ and $\Sigma\omega_i = 1$. Of course, if $\theta = 1$, $v(.) = \ln\phi(.)$ and if $\varepsilon = 1$, $\ln\phi(.) = \sum_{i=1}^m \omega_i \ln c_i$. In the decentralized economy demand functions have constant price elasticity $-\varepsilon$ and unit income elasticity. With this utility function, (8) becomes:

$$\frac{p_i c_i}{c_m} = \left(\frac{\omega_i}{\omega_m}\right)^{\varepsilon}\left(\frac{A_m}{A_i}\right)^{1-\varepsilon} \equiv x_i \qquad \forall i. \tag{10}$$

The new variable x_i is the ratio of consumption expenditure on good i to consumption expenditure on the manufacturing good and will prove useful in the subsequent analysis. We also define consumption expenditure and output per capita in terms of the numeraire:

$$c \equiv \textstyle\sum_{i=1}^m p_i c_i; \qquad y \equiv \sum_{i=1}^m p_i F^i \tag{11}$$

Using static efficiency we derive:

$$c = c_m X; \qquad y = A_m F\left(k,1\right) \tag{12}$$

where $X \equiv \sum_{i=1}^m x_i$. We note that although k is the ratio of the economy-wide capital stock to the labor force, the technology parameter for output is TFP in manufacturing and not an average of all sectors' TFP.

[4] The corresponding transversality condition is $\lim_{t\longrightarrow\infty} k \exp\left(-\int_0^t \left(F_k^m - \delta - \nu\right) d\tau\right) = 0$.

3 Structural change

We define structural change as the state in which at least some of the labor shares are changing over time, i.e., $\dot{n}_i \neq 0$ for at least some i. We derive in the Appendix (Lemma 7) the employment shares:

$$n_i = \frac{x_i}{X}\left(\frac{c}{y}\right) \qquad \forall i \neq m, \tag{13}$$

$$n_m = \frac{x_m}{X}\left(\frac{c}{y}\right) + \left(1 - \frac{c}{y}\right). \tag{14}$$

The first term in the right side of (14) parallels the term in (13) and so represents the employment needed to satisfy the consumption demand for the manufacturing good. The second bracketed term is equal to the savings rate and represents the manufacturing employment needed to satisfy investment demand.

Condition (13) implies that the ratio of employment in sector i to employment in sector j is equal to the ratio x_i/x_j (for $i, j \neq m$). By differentiation we obtain that the growth rate of relative employment depends only on the difference between the sectors' TFP growth rates and the elasticity of substitution between goods:

$$\frac{\dot{n}_i}{n_i} - \frac{\dot{n}_j}{n_j} = (1-\varepsilon)\left(\gamma_j - \gamma_i\right) \qquad \forall i, j \neq m. \tag{15}$$

But (8) implies that the growth rate of the relative price of good i is:

$$\dot{p}_i/p_i = \gamma_m - \gamma_i \qquad \forall i \neq m \tag{16}$$

and so,

$$\frac{\dot{n}_i}{n_i} - \frac{\dot{n}_j}{n_j} = (1-\varepsilon)\left(\frac{\dot{p}_i}{p_i} - \frac{\dot{p}_j}{p_j}\right) \qquad \forall i, j \neq m \tag{17}$$

Proposition 1 *The rate of change of the relative price of good i to good j is equal to the difference between the TFP growth rates of sector j and sector i. In sectors producing only consumption goods, relative employment shares grow in proportion to relative prices, with the factor of proportionality given by one minus the elasticity of substitution across goods.*[5]

[5] All derivations and proofs, unless trivial, are collected in the Appendix.

The dynamics of the individual employment shares satisfy:

$$\frac{\dot{n}_i}{n_i} = \frac{\dot{c/y}}{c/y} + (1-\varepsilon)(\bar{\gamma} - \gamma_i); \qquad \forall i \neq m \tag{18}$$

$$\frac{\dot{n}_m}{n_m} = \left[\frac{\dot{c/y}}{c/y} + (1-\varepsilon)(\bar{\gamma} - \gamma_m)\right]\frac{(c/y)(x_m/X)}{n_m} + \left(\frac{-\dot{c/y}}{(1-c/y)}\right)\left(\frac{1-c/y}{n_m}\right) \tag{19}$$

where $\bar{\gamma} \equiv \sum_{i=1}^{m} (x_i/X)\gamma_i$ is a weighted average of TFP growth rates.[6]

Equation (18) gives the growth rate in the employment share of each consumption sector as a linear function of its own TFP growth rate. The intercept and slope of this function are common across sectors but although the slope is a constant, the intercept is in general a function of time because both c/y and $\bar{\gamma}$ are in general functions of time. Manufacturing, however, does not conform to this rule, because its employment share is made up of two components, one for the production of the consumption good (which behaves similarly to the employment share of consumption sectors) and one for the production of capital goods, which behaves differently.

The properties of structural change follow immediately from (18) and (19). Consider first the case of equality in sectoral TFP growth rates, i.e., let $\gamma_i = \gamma_m\ \forall i$. Our economy in this case is one of balanced TFP growth, with relative prices remaining constant but with many differentiated goods. Because of the constancy of relative prices all consumption goods can be aggregated into one, so we effectively have a two-sector economy, one sector producing consumption goods and one producing capital goods. Structural change can still take place in this economy but only between the aggregate of the consumption sectors and the capital sector, and only if c/y changes over time. If c/y is increasing over time, the savings and investment rate are falling and labor is moving out of the manufacturing sector and into the consumption sectors. Conversely, if c/y is falling over time labor is moving out of the consumption sectors and into manufacturing. In both cases, however, the relative employment shares in consumption sectors are constant.

If c/y is constant over time, structural change requires $\varepsilon \neq 1$ and different rates of sectoral TFP growth rates. It follows immediately from (16), (18) and (19) that if $\dot{c/y} = 0$, $\varepsilon = 1$ implies constant employment shares but changing prices. With constant employment shares faster-growing sectors produce relatively more output

[6]Note that this weighted average is not the average TFP growth rate for the economy as a whole, because the weights take into account only production for consumption purposes, ignoring production for investment purposes.

over time. Price changes in this case are such that consumption demands exactly absorb all the output changes that are due to the different TFP growth rates. But if $\varepsilon \neq 1$, prices still change as before and consumption demands are either too inelastic (in the case $\varepsilon < 1$) to absorb all the output change, or are too elastic ($\varepsilon > 1$) to be satisfied merely by the change in output due to TFP growth. So if $\varepsilon < 1$ employment has to move into the slow-growing sectors and if $\varepsilon > 1$ it has to move into the fast-growing sectors.

Proposition 2 *If $\gamma_i = \gamma_m \ \forall i \neq m$, a necessary and sufficient condition for structural change is $\dot{c}/c \neq \dot{y}/y$. The structural change in this case is between the aggregate of consumption sectors and the manufacturing sector.*

If $\dot{c}/c = \dot{y}/y$, necessary and sufficient conditions for structural change are $\varepsilon \neq 1$ and $\exists i \in \{1, .., m-1\}$ s.t. $\gamma_i \neq \gamma_m$. The structural change in this case is between all sector pairs with different TFP growth rates. If $\varepsilon < 1$ employment moves from the sector with the higher TFP growth rate to the sector with the lower TFP growth rate; conversely if $\varepsilon > 1$.

Proposition 2 for $\varepsilon < 1$ confirms Baumol's (1967; Baumol et al. 1985) claims about structural change. When demand is price inelastic, the sectors with the low productivity growth rate attract a bigger share of labor, despite the rise in their price. The lower the price elasticity, the less the fall in demand that accompanies the price rise, and so the bigger the shift in employment needed to satisfy the high relative consumption. The behavior of the output and consumption shares is obtained from the static efficiency results in (8) and (10):

$$\frac{p_i F^i}{\sum_{i=1}^m p_i F^i} = n_i; \qquad \frac{p_i c_i}{\sum_{i=1}^m p_i c_i} = \frac{x_i}{X}; \quad \forall i. \tag{20}$$

The nominal output shares are equal to the employment shares, so the results obtained for employment shares also hold for them. From (13), nominal consumption shares also exhibit similar dynamic behavior to employment shares, but relative real consumptions satisfy

$$\dot{c}_i/c_i - \dot{c}_j/c_j = \varepsilon \left(\gamma_i - \gamma_j\right); \qquad \forall i, j, \tag{21}$$

an expression also satisfied by real output shares $\forall i, j \neq m$.

A comparison of (15) with (21) reveals that a small ε can reconcile the small changes in the relative real consumption shares with the large changes in relative nominal consumption shares found by Kravis et al. (1983). More recently Sichel (1997) found the same pattern for relative output shares. This finding led Kravis et al. (1983) to conclude that the evidence favored a technological explanation for structural change.

4 Aggregate growth

We now study the aggregate growth path in this economy, with the objective of finding a sufficient set of conditions that satisfy structural change as derived in the preceding section, and in addition satisfy the Kaldor stylized facts of aggregate growth. Recall that for the analysis of structural change we imposed a Hicks-neutral technology. It is well-known that with this type of technology, the economy can be on a steady state only if the production function is Cobb-Douglas. We therefore begin by assuming $F(n_i k_i, n_i) = (n_i k_i)^{\alpha} n_i^{1-\alpha}$, $\alpha \in (0,1)$.[7] With TFP in each sector growing at some rate γ_i, the aggregate economy will also grow at some rate related to the γ_is. The following Proposition derives the evolution of the aggregate economy:

Proposition 3 *Given any initial k_0, the equilibrium of the aggregate economy is defined as a path for the pair $\{c, k\}$ that satisfies the following two differential equations:*

$$\dot{k}/k = A_m k^{\alpha-1} - c/k - (\delta + \nu), \tag{22}$$

$$\theta \dot{c}/c = (\theta - 1)(\gamma_m - \bar{\gamma}) + \alpha A_m k^{\alpha-1} - (\delta + \rho + \nu). \tag{23}$$

Recalling the definition of $\bar{\gamma}$ following equation (19), the key property of our equilibrium is that the contribution of each consumption sector i to aggregate equilibrium is through its weight x_i in $\bar{\gamma}$. Note that because each x_i depends on the sector's relative TFP level (A_i/A_m), the weights here are functions of time.

We define an aggregate balanced growth path such that aggregate output, consumption and capital grow at the same rate. On this path the capital-output ratio k/y is constant, which, by the aggregation in (12), requires $A_m k^{\alpha-1}$ to be constant; i.e., k, and therefore y and c, grow at constant rate $g_m \equiv \gamma_m/(1-\alpha)$, the rate of labor-augmenting technological growth in the capital-producing sector.

[7] Proposition 1 can be modified to allow for different capital shares. Equation (17) remains true but (15) contains an additional term $(1-\varepsilon)(\alpha_j - \alpha_i)\dot{k}_m/k_m$ and (16) contains an additional term $(\alpha_m - \alpha_i)\dot{k}_m/k_m$.

A necessary and sufficient condition for the existence of an aggregate balanced growth path is that the expression $(\theta - 1)(\gamma_m - \bar{\gamma})$ be a constant. To show this, let:

$$(\theta - 1)(\gamma_m - \bar{\gamma}) \equiv \psi \qquad \text{constant.} \tag{24}$$

Define aggregate consumption and the capital-labor ratio in terms of efficiency units, $c_e \equiv cA_m^{-1/(1-\alpha)}$ and $k_e \equiv kA_m^{-1/(1-\alpha)}$. The dynamic equations (22) and (23) become

$$\dot{c}_e/c_e = \left[\alpha k_e^{\alpha-1} + \psi - (\delta + \nu + \rho)\right]/\theta - g_m \tag{25}$$

$$\dot{k}_e = k_e^{\alpha} - c_e - (g_m + \delta + \nu)\, k_e. \tag{26}$$

Equations (25) and (26) parallel the two differential equations in the control and state of the one-sector Ramsey economy, making the aggregate equilibrium of our many-sector economy identical to the equilibrium of the one-sector Ramsey economy when $\psi = 0$, and trivially different from it otherwise. Both models have a saddlepath equilibrium and stationary solutions $\left(\hat{c}_e, \hat{k}_e\right)$ that imply balanced growth in the three aggregates. As anticipated in the aggregate production function (12), a key result is that in our economy the rate of growth of our aggregates in the steady state is equal to the rate of growth of labor-augmenting technological progress in the sector that produces capital goods: the ratio of capital to employment in each sector and aggregate capital per worker grow at rate g_m. When nominal output is deflated by the price of manufacturing goods, output per worker and aggregate consumption per worker also grow at the same rate.

Proposition 2 and the requirement that ψ be constant yield the important Proposition:

Proposition 4 *Necessary and sufficient conditions for the existence of an aggregate balanced growth path with structural change are:*

$$\theta = 1, \tag{27}$$

$$\varepsilon \neq 1; \ \textit{and}\ \exists i \in \{1, .., m\} \ \ s.t.\ \gamma_i \neq \gamma_m.$$

Under the conditions of Proposition 4, $\psi = 0$, and our aggregate economy becomes formally identical to the one-sector Ramsey economy. There are two other conditions that give a constant ψ and so yield balanced aggregate growth: $\gamma_i = \gamma_m\ \forall i$ or $\varepsilon = 1$. But as Proposition 2 demonstrates neither condition permits structural change on the balanced growth path, where c/y is constant.[8]

[8] Proposition 4 brings out the crucial role played by capital goods in aggregate growth, in contrast, for example, to Baumol's claim, who concluded that the economy's growth rate is on an indefinitely declining trend.

Proposition 4 requires the utility function to be logarithmic in the consumption composite ϕ, which implies an intertemporal elasticity of substitution equal to one, but be non-logarithmic across goods, which is needed to yield non-unit price elasticities. A noteworthy implication of Proposition 4 is that balanced aggregate growth does not require constant rates of growth of TFP in any sector other than manufacturing. Because both capital and labor are perfectly mobile across sectors, changes in the TFP growth rates of consumption-producing sectors are reflected in immediate price changes and reallocations of capital and labor across sectors, without effect on the aggregate growth path.

To give intuition for the logarithmic intertemporal utility function we recall that balanced aggregate growth requires that aggregate consumption be a constant fraction of aggregate wealth. With our homothetic utility function this can be satisfied either when the interest rate is constant or when consumption is independent of the interest rate. The relevant interest rate here is the rate of return to capital in consumption units, which is given by the net marginal product of capital in terms of the manufacturing numeraire, $\alpha y/k - \delta$, minus the change in the relative price of the consumption composite, $\gamma_m - \bar{\gamma}$. The latter is not constant during structural change. In the case $\varepsilon < 1$, $\bar{\gamma}$ is falling over time (see Lemma 8 in the Appendix for proof), and so the real interest rate is also falling, and converging to $\alpha y/k - \delta$. With a non-constant interest rate the consumption-wealth ratio is constant only if consumption is independent of the interest rate, which requires a logarithmic utility function.[9]

Our claim that constant growth for the economy's aggregates requires the use of manufacturing price as numeraire, in contrast to the published aggregate series normally studied by macroeconomists, which use some other average price. However, at the level of "stylized facts" there is not much to differentiate growth in our aggregate economy from growth in the more commonly studied one-sector economy. Our aggregate per capita income in (11) is, in nominal terms, $p_m y$. So, if national statistics report real incomes deflated by some other implicit or explicit index $\tilde{p}$, reported real income in our notation is $p_m y/\tilde{p}$. The difference between our aggregate y and the reported one is the ratio of the price of our manufacturing good to the deflator, $p_m/\tilde{p}$. In

[9] In contrast to one-sector models, a constant capital-output ratio in our model does not imply that the rate of return to capital in consumption units is constant. Under our set of restrictions it is mildly decreasing during structural change and converging to a lower bound. After re-examining the data, Barro and Sala-i-Martin (2004, p.13) concluded, consistent with our model, "it seems likely that Kaldor's hypothesis of a roughly stable real rate of return should be replaced by a tendency for returns to fall over some range as an economy develops."

our model, the average relative price of all goods does not grow at precisely constant rate, even on our aggregate balanced growth path, because the relative sector shares that are used to calculate it are changing during structural change. But because sector shares do not change rapidly over time, visually there is virtually nothing to distinguish the "stylized fact" of constant growth in reported per capita GDP with another "stylized fact" of constant growth in our per capita output variable.[10]

Next, we characterize the set of expanding sectors $(\dot{n}_i \geq 0)$, denoted E_t, and the set of contracting sectors $(\dot{n}_i \leq 0)$, denoted D_t, at any time t. We establish

Proposition 5 *Both in the aggregate balanced growth path and in the transition from a low initial capital stock, the set of expanding sectors is contracting over time and the set of contracting sectors is expanding over time:*

$$E_{t'} \subseteq E_t \text{ and } D_t \subseteq D_{t'} \qquad \forall t' > t$$

Asymptotically, the economy converges to an economy with

$$n_m^* = \hat{\sigma} = \alpha \left(\frac{\delta + \nu + g_m}{\delta + \nu + \rho + g_m} \right); \qquad n_l^* = 1 - \hat{\sigma}$$

$\hat{\sigma}$ is the investment rate along the aggregate balanced growth path and sector l denotes the sector with the smallest (largest) TFP growth rate if and only if goods are poor (good) substitutes.

In order to give some intuition for the proof (which is in the Appendix), consider the dynamics of sectors on the aggregate balanced growth path. Along this path, the set of expanding and contracting sectors satisfy:

$$\begin{aligned} E_t &= \{i \in \{1, ..., m\} : (1 - \varepsilon)(\bar{\gamma} - \gamma_i) \geq 0\}; \\ D_t &= \{i \in \{1, ..., m\} : (1 - \varepsilon)(\bar{\gamma} - \gamma_i) \leq 0\}. \end{aligned} \tag{28}$$

Consider the case $\varepsilon < 1$,the one for $\varepsilon > 1$ following by a corresponding argument. For $\varepsilon < 1$, sector i expands if and only if its TFP growth rate is smaller than $\bar{\gamma}$, and contracts if and only if its growth rate exceeds it. But if $\varepsilon < 1$, the weighted average $\bar{\gamma}$ is decreasing over time (see Lemma 8 in the Appendix). Therefore, the set

[10]Kaldor (1961, p.178) spoke of a "steady trend rate" of growth in the "aggregate volume of production." In Ngai and Pissarides (2004, Fig.4) we plot our series of per capital real incomes and the published chain-weighted series for the United States since 1929, and show that they are virtually indistinguishable from each other.

of expanding sectors is shrinking over time, as more sectors' TFP growth rates exceed $\bar{\gamma}$. This feature of the model implies that sectors with intermediate TFP growth rates below the initial $\bar{\gamma}$ exhibit a hump-shaped employment share, an implication that we believe is unique to our model. These employment shares first rises but once $\bar{\gamma}$ drops down to their own γ_i they start to fall.

In contrast to each sector's employment share, once the economy is on the aggregate balanced growth path output and consumption in each consumption sector (as a ratio to the total labor force) grows according to

$$\frac{\dot{F}^i}{F^i} = \frac{\dot{A}_i}{A_i} + \alpha\frac{\dot{k}_i}{k} + \frac{\dot{n}_i}{n_i} = \varepsilon\gamma_i + \alpha g_m + (1-\varepsilon)\,\bar{\gamma}. \tag{29}$$

Thus, if $\varepsilon \leqslant 1$ the rate of growth of consumption and output in each sector is positive, and so sectors never vanish, even though their employment shares in the limit may vanish. If $\varepsilon > 1$ the rate of growth of output may be negative in some low-growth sectors, and since by Lemma 8 $\bar{\gamma}$ is rising over time in this case, their rate of growth remains indefinitely negative until they vanish.

Finally, we examine briefly the implications of $\theta \neq 1$. When $\theta \neq 1$ balanced aggregate growth cannot coexist with structural change, because the term $\psi = (\theta-1)(\gamma_m - \bar{\gamma})$ in the Euler condition (25) is a function of time. But as shown in the Appendix Lemma 8, $\bar{\gamma}$ is monotonic. As $t \to \infty$, ψ converges to the constant $(\theta-1)(\gamma_m - \gamma_l)$, where γ_l is the TFP growth rate in the limiting sector (the slowest or fastest growing consumption sector depending on whether $\varepsilon <$ or > 1). Therefore, the economy with $\theta \neq 1$ converges to an asymptotic steady state with the same growth rate as the economy with $\theta = 1$.

What characterizes the dynamic path of the aggregate economy when $\theta \neq 1$? By differentiation and using Lemma 8 in the Appendix, we obtain

$$\dot{\psi} = (\theta-1)(1-\varepsilon)\textstyle\sum_{i=1}^{m}(x_i/X)\,(\gamma_i - \bar{\gamma})^2 \tag{30}$$

which is of second-order compared with the growth in employment shares in (15), given that the $\gamma's$ are usually small numbers centered around 0.02. Therefore, the rate of growth of the economy during the adjustment to the asymptotic steady state with $\theta \neq 1$ is very close to the constant growth rate of the economy with $\theta = 1$, despite ongoing structural change in both economies.

5 Intermediate goods

Our baseline model has no intermediate inputs and has only one sector producing capital goods. We now generalize it by introducing intermediate inputs and (in the next section) by allowing an arbitrary number of sectors to produce capital goods. The key difference between intermediate goods and capital goods is that capital goods are re-usable while intermediate goods depreciate fully after one usage. The motivation for the introduction of intermediate inputs is that many of the sectors that may be classified as consumption sectors produce in fact for businesses. Business services is one obvious example. Input-output tables show that a large fraction of output in virtually all sectors of the economy is sold to businesses.[11]

As in the baseline model, sectors are of two types. The first type, which consists of sectors such as food and services, produces perishable goods that are either consumed by households or used as intermediate inputs by firms. We continue referring to these sectors as consumption sectors for short. The second type of sector consists of sectors such as engineering and metals and produces goods that can be used as capital. For generality's sake, we assume that the output of the capital-producing sector can also be processed into both consumption goods and intermediate inputs.

The output of consumption sector i is now $c_i + h_i$, where h_i is the output that is used as an intermediate good. Manufacturing output can be consumed, c_m, used as an intermediate input, h_m, or used as new capital, $\dot{k}$. We assume that all intermediate goods h_i are used as an input into an aggregate CES production function $\Phi(h_1, ..., h_m) = \left[\sum_{i=1}^{m} \varphi_i h_i^{(\eta-1)/\eta}\right]^{\eta/(\eta-1)}$ that produces a single intermediate good Φ, with $\eta > 0, \varphi_i \geq 0$ and $\Sigma\varphi_i = 1$. The production functions are modified to $F^i = A_i n_i k_i^{\alpha} q_i^{\beta}$, $\forall i$, where q_i is the ratio of the intermediate good to employment in sector i and β is its input share, with $\alpha, \beta > 0$ and $\alpha + \beta < 1$. When $\beta = 0$, we return to our baseline model. We show in the Appendix that a necessary and sufficient condition for an aggregate balanced growth path with structural change requires $\eta = 1$, i.e. $\Phi(.)$ to be Cobb-Douglas.[12] When $\Phi(.)$ is Cobb-Douglas, our central results from

[11] According to input-output tables for the United States, in 1990 the percentage distribution of the output of two-digit sectors across three types of usage, final consumption demand, intermediate goods and capital goods was 43, 48 and 9 respectively. In virtually all sectors, however, a large fraction of the intermediate goods produced are consumed by the same sector.

[12] Oulton (2001) claims that if there are intermediate goods, and if the elasticity of substitution between the intermediate goods and labor is bigger than 1, Baumol's stagnationist results could be overturned (in the absence of capital). No such possibility arises with Cobb-Douglas production

the baseline model carry through, with some modifications.

The aggregate equilibrium is similar to the one in the baseline model:

$$\dot{c}/c = \alpha A k^{\alpha/(1-\beta)-1} - (\delta + \rho + \nu), \tag{31}$$

$$\dot{k} = (1-\beta) A k^{\alpha/(1-\beta)} - c - (\delta + \nu) k \tag{32}$$

where $A \equiv \left[A_m (\beta \Phi_m)^\beta\right]^{1/(1-\beta)}$ and Φ_m is the marginal product of the manufacturing good in Φ. The growth rate of A is constant and equal to $\gamma = \gamma_m + \left(\beta \sum_{i=1}^m \varphi_i \gamma_i\right) / (1-\beta)$, where φ_i is the input share of sector i in Φ. Therefore, we can define aggregate consumption and the aggregate capital-labor ratio in terms of efficiency units and obtain an aggregate balanced growth path with growth rate $\left(\gamma_m + \beta \sum_{i=1}^m \varphi_i \gamma_i\right) / (1-\alpha-\beta)$, which is the sum of labor-augmenting technological growth rate in the capital-producing sector plus β fraction of the labor-augmenting technological growth rate in all sectors that produce intermediate goods. Recall the aggregate growth rate in the baseline model depended only on the TFP growth rate in manufacturing. In the extended model with intermediate goods, the TFP growth rates in all sectors contribute to aggregate growth, but growth is still constant. If $\beta = 0$ the model collapses to the baseline case.

The employment shares (13) and (14) are now generalize to:

$$n_i = \frac{x_i}{X}\left(\frac{c}{y}\right) + \varphi_i \beta; \qquad \forall i \neq m \tag{33}$$

$$n_m = \left[\frac{x_m}{X}\left(\frac{c}{y}\right) + \varphi_m \beta\right] + \left(1 - \beta - \frac{c}{y}\right). \tag{34}$$

For the consumption sectors, the extra term in (33) captures the employment required for producing intermediate goods. φ_i is the share of sector i's output used for intermediate purposes and β is the share of the aggregate intermediate input in aggregate output. For the manufacturing sector, the terms in the first bracket parallel those of the consumption sectors. The second term captures the employment share for investment purposes.

The relative employment shares across consumption sectors are no longer equal to x_i/x_j (as in the baseline model) because of the presence of intermediate goods. Therefore, Proposition 1 holds for relative prices as in the baseline, but the expression for relative employment needs to be modified. The modification, however, is

functions.

straightforward because $\varphi_i\beta$ is constant, and the results about the direction of structural change hold as in the baseline model. Employment dynamics are now given by,

$$\frac{\dot{n}_i}{n_i - \varphi_i\beta} = \left(\frac{\dot{\overline{c/y}}}{c/y} + (1-\varepsilon)\,\bar{\gamma}\right) - (1-\varepsilon)\gamma_i; \qquad \forall i \neq m. \tag{35}$$

We note that as in the baseline model the right-hand side is made up of a term that is a function of time but is common to all sectors and a second term that is proportional to the sector's own TFP growth rate. When the sector's share of intermediate good production is small the left-hand side is approximately equal to the rate of growth of the sector's employment share. Combining (16) and (35) we obtain the following relation between employment growth and prices

$$\frac{\dot{n}_i}{n_i - \varphi_i\beta} = \left(\frac{\dot{\overline{c/y}}}{c/y} + (1-\varepsilon)\,\bar{\gamma} - \gamma_m\right) + (1-\varepsilon)\frac{\dot{p}_i}{p_i}; \qquad \forall i \neq m, \tag{36}$$

and so (17) now generalizes to:

$$\frac{\dot{n}_i}{n_i - \varphi_i\beta} - \frac{\dot{n}_j}{n_j - \varphi_j\beta} = (1-\varepsilon)\left(\frac{\dot{p}_i}{p_i} - \frac{\dot{p}_j}{p_j}\right) \qquad \forall i, j \neq m. \tag{37}$$

The asymptotic results in Proposition 5 are also modified. Asymptotically, the employment share used for the production of consumption goods still vanishes in all sectors except for the slowest growing one (when $\varepsilon < 1$), but the employment share used to produce intermediate goods, $\varphi_i\beta$, survives in all sectors.

6 Many capital goods

In our second extension we allow an arbitrary number of sectors to produce capital goods. We study this extension with the baseline model without intermediate inputs.

We suppose that there are κ different capital-producing sectors, each supplying the inputs into a production function G, which produces a capital aggregate that can be either consumed or used as an input in all production functions F^i. Thus, the model is the same as before, except that now the capital input k_i is not the output of a single sector but of the production function G. The Appendix derives the equilibrium for the case of a CES function with elasticity μ, i.e., when $G = \left[\sum_{j=1}^{\kappa} \xi_{m_j} \left(F^{m_j}\right)^{(\mu-1)/\mu}\right]^{\mu/(\mu-1)}$, where $\mu > 0$, $\xi_{m_j} \geq 0$ and F^{m_j} is the output of each capital goods sector m_j. G now replaces the output of the "manufacturing" sector in our baseline model, F^m.

It follows immediately that the structural change results derived for the $m-1$ consumption sectors remain intact, as we have made no changes to that part of the model. But there are new results to derive concerning structural change within the capital-producing sectors. The relative employment shares across the capital-producing sectors satisfy:

$$n_{m_j}/n_{m_i} = \left(\xi_{m_j}/\xi_{m_i}\right)^{\mu}\left(A_{m_i}/A_{m_j}\right)^{1-\mu}; \quad \forall i,j=1,..,\kappa \tag{38}$$

$$\frac{\dot{n}_{m_j}}{n_{m_j}}-\frac{\dot{n}_{m_i}}{n_{m_i}} = (1-\mu)\left(\gamma_{m_i}-\gamma_{m_j}\right); \quad \forall i,j=1,..,\kappa \tag{39}$$

If $\mu = 1$ (G is Cobb-Douglas), then the relative employment shares across capital-producing sectors remain constant over time. If $\mu > 1\,(< 1)$, then more productive capital-producing sectors increase (decrease) their employment share over time.

Comparing the new results to the results derived for consumption sectors in the baseline model, the A_m of the baseline model is replaced by $G_{m_j}A_{m_j}$, where G_{m_j} denotes the marginal product and A_{m_j} denotes TFP of capital good m_j. This term measures the rate of return to capital in the jth capital-producing sector, which is equal across all κ sectors because of the free mobility of capital. Defining $A_m \equiv G_{m_1}A_{m_1}$ we derive the growth rate:

$$\gamma_m = \textstyle\sum_{j=1}^{\kappa}\zeta_j\gamma_{m_j}; \quad \zeta_j \equiv \xi_{m_j}^{\mu}A_{m_j}^{(\mu-1)}/\left(\sum_{i=1}^{\kappa}\xi_{m_i}^{\mu}A_{m_i}^{(\mu-1)}\right), \tag{40}$$

which is a weighted average of TFP growth rates in all capital-producing sectors. The dynamic equations for c and k are the same as in the baseline model, given the new definition of γ_m.

If TFP growth rates are equal across all capital-producing sectors, c and k grow at a common rate in the steady state. But then all capital producing sectors can be aggregated into one, and the model reduces to one with a single capital-producing sector. If TFP growth rates are different across the capital-producing sectors and $\mu \neq 1$, there is structural change within the capital-producing sectors along the transition to the asymptotic state. Asymptotically, only one capital-producing sector remains. In the asymptotic state, c and k again grow at common rate, so there exists an asymptotic aggregate balanced growth path with only one capital-producing sector.

A necessary and sufficient condition for the coexistence of an aggregate balanced growth path and multiple capital-producing sectors with different TFP growth rates is $\mu = 1$. The aggregate growth rate in this case is $\gamma_m / (1 - \alpha)$ and (40) implies $\gamma_m = \sum_{j=1}^{\kappa} \xi_{m_j} \gamma_{m_j}$. Using (38), the relative employment shares across capital-producing sectors are equal to their relative input shares in G. There is no structural change within the capital producing sectors, their relative employment shares remaining constant independently of their TFP growth rates.

The extended model with $\varepsilon < 1$ and $\mu = 1$ has clear contrasting predictions about the relation between the dynamics of sectoral employment shares and TFP growth (or relative prices). Sectors that produce primarily consumption goods should exhibit a well-defined linear relation between their employment share growth and their TFP growth rate; sectors that produce many intermediate goods should still have a positive linear relation, but less well-defined, and sectors that produce primarily capital goods should exhibit no linear relation at all between their employment share growth and their relative TFP growth rate.[13]

7 Conclusion

We have shown that predicted sectoral change that is consistent with the facts requires low substitutability between the final goods produced by each sector. Balanced aggregate growth requires in addition a logarithmic intertemporal utility function. Underlying the balanced aggregate growth there is a shift of employment away from sectors with high rate of technological progress towards sectors with low growth, and eventually, in the limit, all employment converges to only two sectors, the sector producing capital goods and the sector with the lowest rate of productivity growth. If the economy also produces intermediate goods the sectors that produce these goods also retain some employment in the limit, for similar reasons.

Our results are consistent with the observation of simultaneous growth in the relative prices and employment shares of stagnant sectors such as personal services, with the near-constancy of real consumption shares when compared with nominal shares, and with the long-run evidence of Kuznets (1966) and Maddison (1980) concerning the decline of agriculture's employment share, the rise and then fall of the manufacturing share and the rise in service share.[14] The key requirement for these results

[13]Preliminary tests reported in Ngai&Pissarides (2004) confirm their ranking.

[14]Kuznets (1966) documented structural change for 13 OECD countries and the USSR between 1800 and 1960 and Maddison (1980) documented the same pattern for 16 OECD countries from 1870 to 1987. They both found a pattern with the same general features as the predictions that we

is again a low substitutability between final goods. Of course, at a finer sector decomposition the elasticity of substitution between two goods may reasonably exceed unity; as for example between the output of the sector producing typewriters and the output of the sector producing word processors. Our model in this case predicts that labor would move from the sector with low TFP growth to the one with the high TFP growth. The approach that we suggested for intermediate and many capital goods, namely the existence of subsectors that produce an aggregate that enters the utility function is an obvious approach to the analysis of these cases. Within the subsectors there is structural change towards the high TFP goods but between the aggregates the flow is from high to low TFP sectors.

We have not undertaken a full empirical test of our model because there are still many features of the data that need to be modeled, as for example, barriers to factor mobility that slow down adjustment, changes in labor supply and trade.[15] However, our baseline model appears to be consistent with the broad facts of growth and structural change, respectively known sometimes as the Kaldor and Kuznets stylized facts of growth.

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obtain when the ranking of the historical TFP growth rates is agriculture-manufacturing-services. Maddison (1980, p. 48) found a "shallow bell shape" for manufacturing employment for each of the 16 OECD countries, which can be reproduced by our model for some intermediate values of the manufacturing TFP growth rate.

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Appendix: Proofs

Lemma 6 *Equations (2), (5) and (7) imply equation (8).*

Proof. Define $f(k) \equiv F(k,1)$, omitting subscript i, (7) implies $F_K = Af'(k)$ and $F_N = A[f(k) - kf'(k)]$. So $F_N/F_K = f(k)/f'(k) - k$, which is strictly increasing in k. Hence, (5) implies $k_i = k_m\ \forall i \neq m$, and together with (2), results follow. ■

Lemma 7 $\forall i \neq m$, n_i *satisfy (13) and (18), and* n_m *satisfy (14), and (19).*

Proof. n_i follows from substituting F^i into (10), and n_m is derived from (2). Given $\dot{x}_i/x_i = (1-\varepsilon)(\gamma_m - \gamma_i)$ and $\dot{X} = \sum_{i=1}^m \dot{x}_i = (1-\varepsilon)(\gamma_m - \bar{\gamma})X$, result follows for $\dot{n}_i/n_i$, $i \neq m$. Using (2), $\dot{n}_m = -\sum_{i\neq m} \dot{n}_i$, so

$$\begin{aligned}
\dot{n}_m &= -\frac{\dot{\overline{c/y}}}{c/y}(1-n_m) - (1-\varepsilon)\left(\frac{c/y}{X}\right)\textstyle\sum_{i\neq m} x_i(\bar{\gamma} - \gamma_i) \\
&= \frac{\dot{\overline{c/y}}}{c/y}\left(\frac{c/y}{X} - \frac{c}{y}\right) + (1-\varepsilon)\left(\frac{c/y}{X}\right)(\bar{\gamma} - \gamma_m),
\end{aligned}$$

so result follows for $\dot{n}_m$. ■

Proposition 3. **Proof.** Use (2) and (8) to rewrite (4) as $\dot{k}/k = A_m k^{\alpha-1}(1 - \sum_{i\neq m} n_i) - c_m/k - (\delta+\nu)$. But $p_i = A_m/A_i$ and by the definition of c, $\dot{k}/k = A_m k^{\alpha-1} - c/k - (\delta+\nu)$. Next, ϕ is homogenous of degree one: $\phi = \sum_{i=1}^m \phi_i c_i = \sum_{i=1}^m p_i c_i \phi_m = \phi_m c$. But $\phi_m = \omega_m (\phi/c_m)^{1/\varepsilon}$ and $c = c_m X$, thus $\phi_m = \omega_m^{\varepsilon/(\varepsilon-1)} X^{1/(\varepsilon-1)}$ and $v_m = \phi^{-\theta}\phi_m = \left(\omega_m^{\varepsilon/(\varepsilon-1)} X^{1/(\varepsilon-1)}\right)^{1-\theta} c^{-\theta}$, so (6) becomes (23). ■

Lemma 8 *$d\bar{\gamma}/dt \lesseqgtr 0 \Leftrightarrow \varepsilon \lesseqgtr 1$.*

Proof. Totally differentiating $\bar{\gamma}$ as defined in Proposition 3

$$\begin{aligned} d\bar{\gamma}/dt &= \textstyle\sum_{i=1}^{m}(x_i/X)\,\gamma_i\,(\dot{x}_i/x_i - \sum_{i=1}^{m}\dot{x}_j/X) \\ &= (1-\varepsilon)\textstyle\sum_{i=1}^{m}(x_i/X)\,\gamma_i\left[\gamma_m - \gamma_i - \sum_{i=1}^{m}(x_i/X)\left(\gamma_m - \gamma_j\right)\right] \\ &= (1-\varepsilon)\left(\bar{\gamma}^2 - \textstyle\sum_{i=1}^{m}(x_i/X)\,\gamma_i^2\right) = -(1-\varepsilon)\textstyle\sum_{i=1}^{m}(x_i/X)\,(\gamma_i - \bar{\gamma})^2. \end{aligned}$$

Since the summation term is always positive the result follows. ■

Proof of Proposition 5

Lemma 9 *Along the aggregate balanced growth path (ABGP), if $\varepsilon \lesseqgtr 1$, n_i is non-monotonic if and only if $\bar{\gamma}_0 \gtreqless \gamma_i$, $\forall i \neq m$. The non-monotonic n_i first increases at a decreasing rate for $t < t_i$, then decreases and converges to constant n_i^* asymptotically, where t_i is such that $\bar{\gamma}_{t_i} = \gamma_i$. The monotonic n_i are decreasing and converge to zero asymptotically. Moreover, define sector s and f such that $\gamma_s = \min\{\gamma_i\}_{i=1,..,m}$ and $\gamma_f = \max\{\gamma_i\}_{i=1,..,m.}$, then $t_s\,(t_f) \to \infty$ if $\varepsilon < (>)\,1$.*

Proof. $\forall i \neq m$, Lemma 7 implies along the ABGP, $\dot{n}_i/n_i = (1-\varepsilon)\left(\bar{\gamma}_t - \gamma_i\right) > 0$ if and only if $\bar{\gamma}_t > \gamma_i$. Lemma 8 implies n_i eventually decreases. So n_i is non-monotonic if and only if $\bar{\gamma}_0 > \gamma_i$.

To establish Proposition 5, assume, without loss of generality, $\varepsilon < 1$, $\gamma_1 > ... > \gamma_{m-1}$ and $\gamma_m > \gamma_{m-1}$. Define sector h s.t. $\gamma_m < \gamma_h \leq \bar{\gamma}_0 < \gamma_{h+1}$ where $1 < h < m-1$. We first prove the results hold along the ABGP. Lemma 9 implies $t_i = 0$ $\forall i \leq h$, and $i \in E_0$ $\forall i \geq h$, moreover, $E_{t_{h+1}} \cup \{h+1\} = E_0$ and $D_{t_h+1} = D_0 \cup \{h+1\}$, thus $E_{t_{h+1}} \subseteq E_0$ and $D_0 \subseteq D_{t_{h+1}}$. Result follows $\forall t > 0$. Next, we prove that the economy converges to a two-sector economy asymptotically. Given $X/x_i = \sum_{i=1}^{m}(\omega_j/\omega_i)^{\varepsilon}(A_i/A_j)^{1-\varepsilon}$, and $A_i/A_j \to 0$ if and only if $\gamma_i < \gamma_j$, so $X/x_s \to 1$. So asymptotically, $n_s^* = \hat{c}_e\hat{k}_e^{-\alpha}$ and $n_m^* = 1 - n_s^*$. We now prove the results hold also in the transition to the ABGP from any small k_0. Let $z \equiv c_e/k_e$, (25) and (26) (with $\psi = 0$ and $\theta = 1$) imply:

$$\dot{z}/z = (\alpha-1)\,k_e^{\alpha-1} + z - \rho, \qquad \dot{k}_e/k_e = k_e^{\alpha-1} - z - (\delta + \nu + g_m)\,.$$

A phase diagram can be drawn with $\dot{z} < 0$ along the transition. For c/y, we have:

$$\frac{\dot{\overline{c/y}}}{c/y} = \frac{\dot{c}_e}{c_e} - \alpha\frac{\dot{k}_e}{k_e} = \alpha z - \rho - (1-\alpha)(\delta + \nu + g_m)\,.$$

Since $\dot{\overline{c/y}} = 0$ along the ABGP but $\dot{z} < 0$ in the transition, thus $\dot{\overline{c/y}} > 0$ and $\ddot{\overline{c/y}} < 0$ in the transition. $\forall t$, $\forall i \neq m$,

$$\dot{n}_i/n_i = \alpha z - \rho - (1-\alpha)(\delta + \nu + g_m) + (1-\varepsilon)(\bar{\gamma} - \gamma_i),$$

which decreases in the transition given lemma 8 and $\dot{z} < 0$. Thus, given any small k_0, if $i \in E_0$ then $\dot{n}_i > 0$, $\ddot{n}_i < 0$, and if $i \neq s$, $i \in E_t$ $\forall t < t_i$, and $i \in D_t$ $\forall t \geq t_i$, where t_i is defined in Lemma 9. If $i \in D_0$, then $i \in D_t$ $\forall t$. So Lemma 9 holds in the transition. ■

Intermediate goods $\forall i$, $F^i \equiv A_i n_i k_i{}^\alpha q_i^\beta$, $\alpha, \beta \in (0,1)$, $\alpha + \beta < 1$. We have

$$F^m = c_m + h_m + (\delta + \nu) k + \dot{k}, \tag{KA}$$

and $F^i = c_i + h_i, \forall i \neq m$. The planner's problem is similar to the baseline with (KA) replacing (4), $\{h_i, c_i, q_i\}_{i=1,..,m}$ as additional controls and $\sum_{i=1}^m n_i q_i = \Phi(h_1, .., h_m)$ as an additional constraint, where Φ is homogenous of degree one, $\Phi_i > 0$ and $\Phi_{ii} < 0$. The static efficiency conditions are:

$$v_i/v_m = F_K^m/F_N^i = F_N^m/F_N^i = F_Q^m/F_Q^i = \Phi_i/\Phi_m; \qquad \forall i, \tag{SE}$$

which implies $k_i = k$, $q_i = \Phi$, $p_i = A_m/A_i$, $\forall i$, $y = A_m k^\alpha \Phi^\beta$, and $\Phi = \sum_{i=1}^m \Phi_i h_i = \sum_{i=1}^m \Phi_m p_i h_i = \Phi_m h$, where $h \equiv \sum_{i=1}^m p_i h_i$. Optimal conditions for h_m and q_m imply $\beta \Phi_m A_m k^\alpha \Phi^{\beta-1} = 1$, so $h = \beta y$ and (KA) becomes $\dot{k} = A_m k^\alpha \Phi^\beta \left(1 - \sum_{i \neq m} n_i\right) - h_m - c_m - (\delta + \nu) k = h(1-\beta)/\beta - c - (\delta + \nu) k$. The dynamic efficiency condition is $-\dot{v}_m/v_m = \alpha A_m k^{\alpha-1} \Phi^\beta - (\delta + \rho + \nu)$, so

$$\dot{c}/c = \alpha h/(\beta k) - (\delta + \rho + \nu), \quad \dot{k}/k = (1-\beta) h/(\beta k) - c/k - (\delta + \nu). \tag{DE}$$

Constant $\dot{c}/c$ requires constant h/k and constant $\dot{k}/k$ requires constant c/k. Thus, $\dot{h}/h$ must be constant. To derive constant $\dot{h}/h$, consider a CES $\Phi = \left(\sum_{i=1}^m \varphi_i h_i^{(\eta-1)/\eta}\right)^{\eta/(\eta-1)}$, then (SE) imply

$$p_i h_i/h_m = (\varphi_i/\varphi_m)^\eta (A_m/A_i)^{1-\eta} \equiv z_i, \quad \forall i. \tag{z$_i$}$$

So $h = Z h_m$, $\Phi_m = \varphi_m^{\eta/(\eta-1)} Z^{1/(\eta-1)}$, and $\Phi = \left(\beta A_m k^\alpha \varphi_m^{\eta/(\eta-1)} Z^{1/(\eta-1)}\right)^{1/(1-\beta)}$, where $Z \equiv \sum_{i=1}^m z_i$. Hence, $h = \Phi/\Phi_m = (\beta A_m k^\alpha)^{1/(1-\beta)} \left(\varphi_m^{\eta/(\eta-1)} Z^{1/(\eta-1)}\right)^{\beta/(1-\beta)}$, and so

$$(1-\beta)\dot{h}/h = \left(\gamma_m + \alpha \dot{k}/k\right) + \beta\left(\sum_{i=1}^m (z_i/Z)\gamma_i - \gamma_m\right),$$

which is constant if $\sum_{i=1}^{m} z_i\gamma_i$ is constant. Given γ_i are not the same across all i, using (z_i), constancy requires $\eta = 1$, and so $\Phi = \prod_{i=1}^{m} h_i^{\varphi_i}$, $Z = 1/\varphi_m$, and $z_i = \varphi_i/\varphi_m$, $\forall i$. (SE) imply $\Phi = h_m \prod_{i=1}^{m} (z_i A_i/A_m)^{\varphi_i}$ and so $\Phi_m = \varphi_m \Phi/h_m = \prod_{i=1}^{m} (\varphi_i A_i/A_m)^{\varphi_i}$. But $\Phi = [\beta A_m k^\alpha \Phi_m]^{1/(1-\beta)}$, so $h = \Phi/\Phi_m = (\beta A_m k^\alpha)^{1/(1-\beta)} \Phi_m^{\beta/(1-\beta)}$. (DE) becomes:

$$\dot{c}/c + \delta + \rho + \nu = \alpha A k^{\alpha/(1-\beta)-1}; \qquad \dot{k} + c + (\delta + \nu) k = (1-\beta) A k^{\alpha/(1-\beta)},$$

where $A \equiv \left[A_m (\beta\Phi_m)^\beta\right]^{1/(1-\beta)}$. Define $c_e \equiv cA^{-(1-\beta)/(1-\alpha-\beta)}$, $k_e \equiv kA^{-(1-\beta)/(1-\alpha-\beta)}$, and $\gamma \equiv \dot{A}/A = \left[\gamma_m + \beta \sum_{i=1}^{m} \varphi_i (\gamma_i - \gamma_m)\right] / (1-\beta) = \gamma_m + \left(\beta \sum_{i=1}^{m} \varphi_i\gamma_i\right) / (1-\beta)$,

$$\dot{c}_e/c_e = \alpha k_e^{\alpha/(1-\beta)-1} - (\delta + \rho + \nu + g); \qquad \dot{k}_e = (1-\beta) k_e^{\alpha/(1-\beta)} - c_e - (\delta + \nu + g) k_e,$$

which imply existence and uniqueness of an ABGP. The growth rate is $g \equiv (1-\beta)\gamma/(1-\alpha-\beta) = \left(\gamma_m + \beta \sum_{i=1}^{m} \varphi_i\gamma_i\right) / (1-\alpha-\beta)$. We obtain n_i using $F^i = c_i + h_i$, $\forall i \neq m$, i.e. $A_i n_i k^\alpha \Phi^\beta p_i = p_i (c_i + h_i) = x_i c_m + z_i h_m = cx_i/X + \varphi_i h$. Substitute p_i and h to obtain $n_i y = cx_i/X + \varphi_i \beta y$, finally obtain (33) and (34).

Many capital-producing sectors $\forall j$, $F^{m_j} \equiv A_{m_j} n_{m_j} k_{m_j}^\alpha$, which together produce good m through $G = \left[\sum_{j=1}^{\kappa} \xi_{m_j} (F^{m_j})^{(\mu-1)/\mu}\right]^{\mu/(\mu-1)}$, $\xi_{m_j} \geqslant 0, \mu > 0$, and $\sum_{j=1}^{\kappa} \xi_{m_j} = 1$. The planner's problem is similar to the baseline model with $\dot{k} = G - c_m - (\delta + \nu) k$ replacing (4), and $\left(k_{m_j}, n_{m_j}\right)_{j=1,..,\kappa}$ as additional controls.

The static efficiency conditions are $F_K^i/F_N^i = F_K^{m_j}/F_N^{m_j}$, $\forall i \neq m$, $\forall j$, so $k_i = k_{m_j} = k$. Also $G_{m_j}/G_{m_i} = F_K^{m_i}/F_K^{m_j} = A_{m_i}/A_{m_j}$, $\forall i, j$, which implies $n_{m_j}/n_{m_i} = \left(\xi_{m_j}/\xi_{m_i}\right)^\mu \left(A_{m_i}/A_{m_j}\right)^{1-\mu}$ and grows at rate $(1-\mu)\left(\gamma_{m_i} - \gamma_{mj}\right)$. Let $n_m \equiv \sum_{j=1}^{\kappa} n_{m_j}$, we have $n_m = n_{m_1} \sum_{j=1}^{\kappa} \left(\xi_{m_j}/\xi_{m_1}\right)^\mu \left(A_{m_1}/A_{m_j}\right)^{1-\mu}$. Next, $p_i = v_i/v_m = A_m/A_i$, $\forall i \neq m$, where $A_m \equiv G_{m_1} A_{m_1}$. Thus, n_i/n_j and p_i/p_j are the same as in the baseline.

To derive the aggregate equilibrium, note that $G = \sum_{j=1}^{\kappa} F^{m_j} G_{m_j} = A_m k^\alpha n_m$, so $\dot{c}/c$ and $\dot{k}/k$ are the same as the baseline, so the equilibrium is the same as the baseline if $\gamma_m \equiv \dot{A}_m/A_m$ is constant, which we now derive. Given $G_{m_1} = \xi_{m_1} (G/F^{m_1})^{1/\mu}$ and $G/F^{m_1} = \left[\sum_{j=1}^{\kappa} \xi_{m_j} \left(A_{m_j} n_{m_j}/(A_{m_1} n_{m_1})\right)^{(\mu-1)/\mu}\right]^{\mu/(\mu-1)}$, using the result on n_{m_j}/n_{m_1} we have $G/F^{m_1} = \left[\sum_{j=1}^{\kappa} \xi_{m_j}^\mu \left(\xi_{m_1} A_{m_1}\right)^{1-\mu} A_{m_j}^{(\mu-1)}\right]^{\mu/(\mu-1)}$, thus $A_m = G_{m_1} A_{m_1} = \left[\sum_{j=1}^{\kappa} \xi_{m_j}^\mu A_{m_j}^{(\mu-1)}\right]^{1/(\mu-1)}$ and $\gamma_m = \sum_{j=1}^{\kappa} \zeta_j \gamma_{m_j}$, where $\zeta_j \equiv \xi_{m_j}^\mu A_{m_j}^{(\mu-1)} / \left(\sum_{j=1}^{\kappa} \xi_{m_j}^\mu A_{m_j}^{(\mu-1)}\right)$. So γ_m is constant if $(\mu-1) \sum_{j=1}^{\kappa} \zeta_j \left(\gamma_{m_j} - \gamma_m\right)^2 = 0$, i.e. if (1) $\gamma_{m_i} = \gamma_{m_j}$, $\forall i, j$, or (2) $\mu = 1$. If (1) is true, the model reduces to only one capital-producing sector. Thus, coexistence of multiple capital-producing sectors and an ABGP requires (2), i.e., $G = \prod_{j=1}^{\kappa} (F^{m_j})^{\xi_j}$ and $\gamma_m = \sum_{j=1}^{\kappa} \xi_{m_j} \gamma_{m_j}$.