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Semiparametric Estimation with Generated Covariates

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ABSTRACT

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In this paper, we study a general class of semiparametric optimization estimators of a vector-valued parameter. The criterion function depends on two types of infinite-dimensional nuisance parameters: a conditional expectation function that has been estimated nonparametrically using generated covariates, and another estimated function that is used to compute the generated covariates in the first place. We study the asymptotic properties of estimators in this class, which is a nonstandard problem due to the presence of generated covariates. We give conditions under which estimators are root-*n* consistent and asymptotically normal, and derive a general formula for the asymptotic variance.

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1. Introduction

In this paper, we study a general class of semiparametric optimization estimators of a vector-valued parameter. The criterion function depends on two types of infinite-dimensional nuisance parameters: a conditional expectation function that has been estimated nonparametrically using generated covariates, and another estimated function that is used to obtain the generated covariates in the first place. The nonparametric component may be profiled and thus depend on unknown finite-dimensional parameters. Generated covariates may originate from an either parametric, semiparametric or nonparametric first step. Deriving asymptotic properties of estimators in this class is a non-standard problem due to the presence of generated covariates. We give conditions under which estimators are root-n consistent and asymptotically normal, and derive a general formula for the asymptotic variance. We also apply our methods to two substantial examples: estimation of average treatment effects via regression on the propensity score (Rosenbaum and Rubin, 1983), and estimation of production functions in the presence of serially correlated technology shocks (Olley and Pakes, 1996; Levinsohn and Petrin, 2003). In both cases, our results contribute new insights to the existing literature.

Semiparametric estimation problems involving both finite- and infinite-dimensional parameters are central to econometrics, and are studied extensively under general conditions by e.g. Newey (1994), Andrews (1994), Chen and Shen (1998), Ai and Chen (2003, 2007), Chen, Linton, and Van Keilegom (2003), Chen and Pouzo (2009), or Ichimura and Lee (2010). None of these papers explicitly considers the case of generated covariates in the nonparametric component. However, as we argue in this paper, it turns out that in order to account for such a structure in semiparametric models it is not necessary to derive a completely new theory. Perhaps surprisingly, the "high-level" conditions given in the aforementioned papers are mostly sufficiently general to encompass the generation step, and only the methods used to verify them need to be adapted. Compared to a standard analysis, the main difficulties occur when establishing a uniform rate of consistency for the nonparametric component (e.g. Newey, 1994, Assumption 5.1(ii); or Chen, Linton, and Van Keilegom, 2003, Condition (2.4)), and an asymptotic normality result for a linearized version of the objective function (e.g. Newey, 1994, Assumption 5.3 and

Lemma 1; or Chen, Linton, and Van Keilegom, 2003, Condition (2.6)).

The main contribution of our paper is to provide a connection between the extensive literature on estimation and inference in semiparametric models and the one on applications with generated covariates. We derive a new stochastic expansion that characterizes the influence of generated covariates in the model's nonparametric component on the asymptotic properties of the final estimator. We then show how to directly apply this expansion to verify the above-mentioned uniform consistency and asymptotic normality conditions. The expansion, which is proven using techiques from empirical process theory (e.g. Van der Vaart and Wellner, 1996; van de Geer, 2009), is related to a result in Mammen, Rothe, and Schienle (2011) for purely nonparametric regression problems with generated covariates. The main difference is that in the present paper we derive bounds on weighted integrals of the remainder term instead of controlling its supremum norm. This requires substantially different mathematical methods. The new bounds shrink at a considerably faster rate than those obtained in Mammen, Rothe, and Schienle (2011), which is critical for our development of a general theory of semiparametric estimation with generated covariates.

As a further contribution, we provide an explicit formula for the asymptotic variance of semiparametric estimators contained in the general class we consider. Compared to an infeasible procedure that uses the true values of the covariates, the influence function of such an estimator generally contains two additional terms: one that accounts for using generated covariates to estimate the nonparametric component, and one that accounts for the direct influence of generated covariates in other parts of the model, e.g. through determining the point of evaluation of the infinite-dimensional parameter. As a byproduct, we obtain a characterization of cases under which these two adjustment terms exactly offset each other, and thus do not affect first-order asymptotic theory. Our methods can also be used to verify conditions under which a bootstrap procedure leads to asymptotically valid inference. The latter aspect can be important in many applications where the asymptotic variance is difficult to estimate.

Our paper is related to an extensive literature on models with generated covariates. To the best of our knowledge, Newey (1984) and Murphy and Topel (1985) were among the first to study the theoretical properties of such two-step estimators in a fully parametric setting. Pagan (1984) and Oxley and McAleer (1993) provide extensive surveys. Nonparametric regression with (possibly nonparametrically) generated covariates is studied by Mammen, Rothe, and Schienle (2011) under general conditions. See their references for a list of examples, and Andrews (1995), Song (2008) and Sperlich (2009) for related results. Examples of semiparametric applications with generated covariates include Olley and Pakes (1996), Heckman, Ichimura, and Todd (1998), Li and Wooldridge (2002), Levinsohn and Petrin (2003), Blundell and Powell (2004), Linton, Sperlich, and Van Keilegom (2008), Rothe (2009) and Escanciano, Jacho-Chávez, and Lewbel (2010), among many others. Hahn and Ridder (2011) use Newey's (1994) path-derivative method to derive the form of the influence function of semiparametric linear, just-identified GMM-type estimators in the presence of generated covariates, but do not study conditions for the estimators' $\sqrt{n}$-consistency or asymptotic normality. Our paper complements and extends their findings by deriving such conditions for a larger class of semiparametric models, allowing e.g. for profiled optimization estimators. We also derive a formula for the asymptotic variance of estimators in this more general class, which does not involve functional derivatives, and discuss validity of the bootstrap for inference. Escanciano, Jacho-Chávez, and Lewbel (2011) provide stochastic expansions for sample means of weighted semiparametric regression residuals with potentially generated regressors in a particular class of "index models", which is contained in the general class we study in this paper. Their approach also relies on certain high-level conditions that seem to be difficult to verify in practice. Our results use direct bounds to control the impact of generated covariates, and apply to a wider range of applications. We discuss the relationship between Hahn and Ridder (2011), Escanciano, Jacho-Chávez, and Lewbel (2011), and the results in our paper in more detail in Section 4.4.

The remainder of the paper is structured as follows: In Section 2, we describe the class of models we consider. In Section 3, we present our main technical result, a stochastic expansion that characterizes the influence of generated covariates in the model's nonparametric component. Section 4 shows how this expansion can be used to verify classic conditions for $\sqrt{n}$-consistency and asymptotic normality of semiparametric estimators, and derives a general formula for the asymptotic variance. In Section 5, we discuss two econometric applications that make use of our results. All proofs and further details on the applications are collected in Appendix A and B, respectively.

2. Generated Covariates in Semiparametric Models

We consider a general class of semiparametric optimization estimators where the criterion function depends on two types of infinite dimensional nuisance parameters: a conditional expectation function that has been estimated nonparametrically using generated covariates, and another estimated function that is used to compute the generated covariates in a first step. No specific estimation procedure is required for the latter object. Our results cover both parametrically and nonparametrically generated covariates, as well as intermediate cases. The setting and notation is otherwise similar to Chen, Linton, and Van Keilegom (2003), and thus allows for nonsmooth criterion functions and profiled estimation of the nonparametric components.

2.1. Model and Estimation Procedure. Let $Z = (Y, X, W) \in \mathbb{R}^{d_Z}$ be a random variable distributed according to some probability measure P_0 that is contained in a semiparametric model $\mathcal{P} = \{P_{\theta,\xi} : \theta \in \Theta, \xi \in \Xi\}$, where $\Theta \subset \mathbb{R}^{d_\theta}$ denotes a finite dimensional parameter space with generic element θ, and $\Xi = \mathcal{M} \times \mathcal{R}$ is an infinite dimensional parameter space with generic element $\xi = (m, r)$. Denote by $\theta_0 \in \Theta$ and $\xi_0(\cdot, \theta) = (m_0(\cdot, \theta), r_0(\cdot)) \in \Xi$ the true values of the finite and infinite dimensional parameter, respectively, which implies that $P_0 = P_{\theta_0,\xi_0(\cdot,\theta_0)}$. We assume that there exists a nonrandom function $q : \text{supp}(Z)\times\Theta\times\Xi \to \mathbb{R}^{d_q}$ such that $Q(\theta, \xi_0(\cdot, \theta)) = \mathbb{E}(q(Z, \theta, \xi_0(\cdot, \theta))) = 0$ if and only if $\theta = \theta_0$. The parametric component of our semiparametric model is thus identified via a moment condition. For simplicity, we also assume that for every $\xi \in \Xi$ the objective function $Q(\theta, \xi(\cdot, \theta))$ depends on the nuisance parameter ξ through its value over some compact set $I_T^* \times I_R^*$ only, which is useful to later accommodate "fixed trimming" schemes into the estimation procedure.

We also impose certain restrictions on the nature of the infinite dimensional parameter $\xi_0(\cdot, \theta) = (m_0(\cdot, \theta), r_0(\cdot))$. First, we assume that r_0 is identified from the distribution of $W \subset Z$, and that this distribution does not depend on the true value of the other parameters in the model. This allows for a consistent estimate of r_0 to be computed without knowledge of θ_0 and m_0. Second, we assume that $m_0(\cdot, \theta)$ is a conditional expectation function that depends on $\theta \in \Theta$ and the true value r_0 through the relationship $m_0(\cdot, \theta) = \mathbb{E}(Y|T(X, \theta, r_0) = \cdot)$ where $T(X, \theta, r) = t(X, r(X_r), \theta)$ is a random vector of dimension d_T, $X_r \subset X$ are the covariates that enter the function r, and $t : \mathbb{R}^{d_X} \times \mathbb{R}^{d_r} \times \Theta \to \mathbb{R}^{d_T}$ is a known function. The role of r_0 is thus to generate (some of) the covariates used to compute the function m_0. By allowing m_0 to depend on X and $r_0(X_r)$ through a known transformation indexed by θ, our setup includes a broad class of index models that require profiling of the nonparametric component.

To make the notation more compact, we usually suppress the arguments of the infinite dimensional parameters, writing $(\theta, \xi) = (\theta, m, r) \equiv (\theta, m(\cdot, \theta), r(\cdot))$, $(\theta, \xi_0) = (\theta, m_0, r_0) \equiv (\theta, m_0(\cdot, \theta), r_0(\cdot))$, and $(\theta_0, \xi_0) = (\theta_0, m_0, r_0) \equiv (\theta_0, m_0(\cdot, \theta_0), r_0(\cdot))$. We also write $T(\theta, r) \equiv T(X, \theta, r)$, $T(\theta) \equiv T(\theta, r_0)$, $T(r) \equiv T(\theta_0, r)$ and $T \equiv T(\theta_0, r_0)$. We assume that Ξ is a class of continuous and bounded functions endowed with the pseudo-norm $\|\cdot\|_\Xi$ induced by the sup-norm, i.e. we have $\|\xi\|_\Xi = \sup_\theta \sup_x |m(x, \theta)| + \sup_{x_r} |r(x_r)|$. We also write $\|B\| = (\mathrm{tr}(B'AB))^{1/2}$ for any matrix B, where we suppress the dependence of the norm on the fixed symmetric positive definite matrix A for notational convenience.

Given an i.i.d. sample $(Z_1, \ldots, Z_n)$ from the distribution of Z, a three-step semiparametric extremum estimator $\hat{\theta}$ of θ_0 can be constructed as follows. In the first step, we compute a (possibly nonparametric) estimate $\hat{r}$ of r_0. In the second step, for every $\theta \in \Theta$ we obtain an estimate $\widehat{m}(\cdot, \theta)$ of $m_0(\cdot, \theta)$ through a nonparametric regression of Y on the generated covariates $\hat{T}(\theta) = T(\theta, \hat{r})$. We discuss how to implement these two estimation procedures in detail below. Finally, writing $(\theta, \hat{\xi}) = (\theta, \widehat{m}(\cdot, \theta), \widehat{r}(\cdot))$, we define the estimator $\hat{\theta}$ of θ_0 as any approximate solution to the problem of minimizing a semiparametric GMM-type objective function:

$$\|Q_n(\hat{\theta}, \hat{\xi})\| = \inf_{\theta \in \Theta} \|Q_n(\theta, \hat{\xi})\| + o_p(1/\sqrt{n}), \tag{2.1}$$

where $Q_n(\theta, \hat{\xi}) = \frac{1}{n} \sum_{i=1}^n q(Z_i, \theta, \hat{\xi})$. Here, we avoid evaluating $\hat{\xi}$ in areas where it is imprecisely estimated by restricting the influence of the nuisance parameter to be exceeded through its value over some compact set $I_T^* \times I_R^*$ introduced above. Such "fixed trimming" procedures are commonly used to derive properties of profiled semiparametric estimators.

Our estimator is a semiparametric procedure involving generated covariates, in the sense that a preliminary estimate $\hat{r}$ of the nuisance parameter r_0 is used to compute the covariates entering the nonparametric regression procedure to estimate $m_0(\cdot,\theta)$. Note that because $\hat{r}$ is also allowed to appear as a separate argument in the objective function Q_n, it does not only determine the shape of the function $\widehat{m}$, but could also exert a direct influence. For instance, the function $\widehat{m}$ can be evaluated at (some transformation of) the generated covariates. This flexibility is required for all examples we consider below.

For the later asymptotic analysis, it will be useful to also consider an infeasible estimation procedure that uses the true value r_0 instead of an estimate $\hat{r}$. Such an estimator $\tilde{\theta}$ of θ_0 can be obtained by first computing an estimate $\widetilde{m}(\cdot,\theta)$ of $m_0(\cdot,\theta)$ via nonparametric regression of Y on $T(\theta)$ for every $\theta \in \Theta$, and then finding an approximate minimizer of an infeasible version of the objective function:

$$\|Q_n(\tilde{\theta},\hat{\xi})\| = \inf_{\theta\in\Theta} \|Q_n(\theta,\tilde{\xi})\| + o_p(1/\sqrt{n}) \tag{2.2}$$

where $(\theta,\tilde{\xi}) = (\theta,\tilde{m}(\cdot,\theta),r_0(\cdot))$. In order to distinguish the two procedures, we refer to $\hat{\theta}$ and $\widehat{m}$ in the following as the *real* estimators of θ_0 and m_0, respectively, and to $\tilde{\theta}$ and $\widetilde{m}$ as the corresponding *oracle* estimators.

2.2. A Framework for Asymptotic Analysis. It is straightforward to show that $\hat{\theta}$ is a consistent estimate of the true value θ_0 under standard conditions. We therefore focus on the more interesting problem of establishing its asymptotic distribution. A number of papers have given "high level" conditions for semiparametric estimators to be root-n consistent and asymptotically normal in models that *do not* involve generated covariates. Examples include Newey (1994), Andrews (1994), Chen and Shen (1998), Ai and Chen (2003), Chen, Linton, and Van Keilegom (2003), or Ichimura and Lee (2010). It turns out that these conditions are generally sufficient to establish the same type of asymptotic properties for semiparametric estimators in models *with* generated covariates. What needs to be adjusted, however, are the arguments to verify some of them.

To illustrate how previous results in the literature on semiparametric estimation can be adapted to our context, consider the main theorem from Chen, Linton, and Van Keilegom (2003).[1] Before we repeat their result, we have to introduce some further notation. Since

[1]A similar argument could be made for the respective results in one of the other papers mentioned above.

we assume that $\hat{\theta}$ is consistent, we can work with small subsets of the parameter spaces. For some small $\delta > 0$, define $\Theta_\delta = \{\theta \in \Theta : \|\theta - \theta_0\| \leq \delta\}$ and $\Xi_\delta = \{\xi \in \Xi : \|\xi - \xi_0\|_\Xi \leq \delta\}$. Furthermore, for any $(\theta, \xi) \in \Theta \times \Xi$, we denote the ordinary derivative of $Q(\theta, \xi)$ with respect to θ by $Q^\theta(\theta, \xi)$. For any $\theta \in \Theta$, we say that $Q(\theta, \xi)$ is pathwise differentiable at $\xi \in \Xi$ in the direction $\bar{\xi}$ if there exists a continuous linear functional $Q^\xi(\theta, \xi) : \Theta \times \Xi \to \mathbb{R}^l$ such that $Q^\xi(\theta, \xi)[\bar{\xi}] = \lim_{\tau \to 0}(Q(\theta, \xi + \tau\bar{\xi}) - Q(\theta, \xi))/\tau$. The functional $Q^\xi(\theta, \xi)$ is called the pathwise derivative of $Q(\theta, \xi)$.

Theorem 1 (Chen, Linton, and Van Keilegom (2003))**.** *Suppose that* $\theta_0 \in int(\Theta)$ *satisfies* $Q(\theta_0, \xi_0) = 0$*, that* $\hat{\theta} = \theta_0 + o_p(1)$*, and that:*

(N1) $\|Q_n(\hat{\theta}, \hat{\xi})\| = \inf_{\theta \in \Theta} \|Q_n(\theta, \hat{\xi})\| + o_p(1/\sqrt{n})$.

(N2) (i) the ordinary derivative $Q^\theta(\theta, \xi_0)$ *of* $Q(\theta, \xi_0)$ *in* θ *exists for* $\theta \in \Theta_\delta$ *and is continuous at* $\theta = \theta_0$*; (ii) the matrix* $Q^\theta_0 = Q^\theta(\theta_0, \xi_0)$ *is of full rank.*

(N3) For all $\theta \in \Theta_\delta$ *the pathwise derivative* $Q^\xi(\theta, \xi_0)[\xi - \xi_0]$ *of* $Q(\theta, \xi_0)$ *exists in all directions* $[[\xi - \xi_0]] \in \Xi$*; and for all* $(\theta, \xi) \in \Theta_{\delta_n} \times \Xi_{\delta_n}$ *with a positive sequence* $\delta_n = o(1)$*: (i)* $\|Q(\theta, \xi) - Q(\theta, \xi_0) - Q^\xi(\theta, \xi_0)[\xi - \xi_0]\| \leq c\|\xi - \xi_0\|^2_\Xi$ *for a constant* $c \geq 0$*; (ii)* $\|Q^\xi(\theta, \xi_0)[\xi - \xi_0] - Q^\xi_0[\xi - \xi_0]\| \leq o(1)\delta_n$*, where* $Q^\xi_0[\xi - \xi_0] = Q^\xi(\theta_0, \xi_0)[\xi - \xi_0]$.

(N4) $\hat{\xi} \in \Xi$ *with probability tending to one; and* $\|\hat{\xi} - \xi_0\|_\Xi = o_p(n^{-1/4})$

(N5) For any positive sequence $\delta_n = o(1)$.

$$\sup_{\|\theta - \theta_0\| \leq \delta_n, \|\xi - \xi_0\|_\Xi \leq \delta_n} \frac{\sqrt{n}\|Q_n(\theta, \xi) - Q(\theta, \xi) - Q_n(\theta_0, \xi_0)\|}{1 + \sqrt{n}(\|Q_n(\theta, \xi)\| + \|Q(\theta, \xi)\|)} = o_p(1)$$

(N6) $\sqrt{n}(Q_n(\theta_0, \xi_0) + Q^\xi_0[\hat{\xi} - \xi_0]) \xrightarrow{d} N(0, V)$ *for some finite matrix* V.

Then $\sqrt{n}(\hat{\theta} - \theta_0) \xrightarrow{d} N(0, \Omega)$*, where* $\Omega = (Q^{\theta\mathsf{T}}_0 A Q^\theta_0)^{-1} Q^{\theta\mathsf{T}}_0 A V A Q^\theta_0 (Q^{\theta\mathsf{T}}_0 A Q^\theta_0)^{-1}$.

Chen, Linton, and Van Keilegom (2003) provide an extensive discussion of the conditions of Theorem 1, arguing that they are fairly general and thus satisfied in a wide range of semiparametric models. Moreover, the result is sufficiently flexible to apply in our setting. Neither of its conditions nor one of the steps in its proof rules out the type of semiparametric estimation problems with generated covariates we consider in this paper. Asymptotic normality of the real estimator of $\hat{\theta}$ can thus simply be established by checking (N1)–(N6). There is no need to develop a completely new theory.[2]

This does not imply that the presence of generated covariates does not affect the asymptotic properties of our estimator. Verification of the "uniform convergence" condition (N4) and the "asymptotic normality" condition (N6) are substantially more complicated, and the asymptotic variance V in (N6) will generally be different from the one we would have obtained if the true value r_0 had been used in the estimation procedure instead of the estimate $\hat{r}$. In the following section, we therefore derive new and general methods to check conditions like (N4) and (N6). On the other hand, note that the remaining conditions of Theorem 1 are not affected by the presence of generated covariates, and can thus be verified by standard arguments: (N1) simply states that $\hat{\theta}$ is an approximate minimizer of the objective function, which we assumed in the first place; (N2) and (N3) are smoothness conditions on the population moment function, and (N5) is a stochastic equicontinuity condition. Neither involves estimates of the nonparametric components of our model, and thus they can be verfied independently of the issue of generated covariates.

3. Controlling the Influence of Generated Covariates

This section contains our main technical result. In particular, we consider a stochastic expansion of nonparametrically estimated regression functions under very general conditions, deriving a sharp bound on weighted averages of the respective remainder terms. This is the key ingredient for showing condition (N6). Throughout this section, we use the notation that for any vector $a \in \mathbb{R}^d$ the values $a_{min} = \min_{1 \leq j \leq d} a_j$ and $a_{max} = \max_{1 \leq j \leq d} a_j$ denote the smallest and largest of its elements, respectively, $a_+ = \sum_{j=1}^d a_j$ denotes the sum of its elements, $a_{-k} = (a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_d)$ denotes the $d-1$-dimensional subvector of a with the kth element removed, and $a^b = (a_1^{b_1}, \ldots, a_d^{b_d})$ for any vector $b \in \mathbb{R}^d$.

[2]To the best of our knowledge, this point has not been made explicitly in the literature on semiparametric estimation. However, it has at least implicitly been noted for a special case in Linton, Sperlich, and Van Keilegom (2008).

3.1. Assumptions. To derive our main result, we need to be more specific about the estimation procedures for the infinite-dimensional nuisance parameters. We do not require a specific procedure for the estimator $\hat{r}$ of r_0, but only impose certain "high-level" restrictions that cover a wide range of methods. Given an estimate of r_0, for every $\theta \in \Theta$ we then obtain an estimate of $m_0(\cdot, \theta)$ through a nonparametric regression of Y on the generated covariates $\hat{T}(\theta) = t(X, \hat{r}(X_r), \theta)$ using p-th order local polynomial smoothing. Our estimator is thus given by $\widehat{m}(x, \theta) = \widehat{\alpha}$, where

$$(\widehat{\alpha}, \widehat{\beta}) = \underset{\alpha,\beta}{\operatorname{argmin}} \sum_{i=1}^{n} (Y_i - \alpha - \sum_{1 \leq u_+ \leq p} \beta_u^\mathsf{T} (\widehat{T}_i(\theta) - x)^u)^2 K_h(\widehat{T}_i(\theta) - x) \ , \tag{3.1}$$

where $K_h(v) = \prod_{j=1}^{d_T} \mathcal{K}(v_j/h_j)/h_j$ is a d-dimensional product kernel built from the univariate kernel function $\mathcal{K}$, $h = (h_1, ..., h_{d_T})$ is a vector of bandwidths that tend to zero as the sample size n tends to infinity, and $\sum_{1 \leq u_+ \leq p}$ denotes the summation over all $u = (u_1, \ldots, u_p)$ with $1 \leq u_+ \leq d_T$. For $p = 1$, we get the usual local linear estimator. We allow for uneven orders $p > 1$ for the purpose of bias control. To present our results later, it will also be useful to introduce the infeasible oracle estimate $\widetilde{m}(\cdot, \theta)$, which is obtained via local linear smoothing of Y versus $T(\theta)$ for every $\theta \in \Theta$, i.e. it is given by $\widetilde{m}(x, \theta) = \widetilde{\alpha}$, where

$$(\widetilde{\alpha}, \widetilde{\beta}) = \underset{\alpha,\beta}{\operatorname{argmin}} \sum_{i=1}^{n} (Y_i - \alpha - \sum_{1 \leq u_+ \leq p} \beta_u^\mathsf{T} (T_i(\theta) - x)^u)^2 K_h(T_i(\theta) - x).$$

We focus on local polynomial estimation for $m_0(\cdot, \theta)$ in this paper because the particular structure of the estimator facilitates controlling the presence of generated covariates (see Mammen, Rothe, and Schienle, 2011), and does not require a separate treatment of boundary regions. While it might be possible to conduct a similar analysis for other nonparametric procedures, such as e.g. orthogonal series estimators, we conjecture that this would require substantially more involved technical arguments.

Assumption 1 (Regularity). *We assume the following properties for the data distribution, the bandwidth, and kernel function $\mathcal{K}$.*

(i) The sample observations Z_i are independent and identically distributed.

(ii) The parameter space Θ is compact. For every $\theta \in \Theta$, the random vector $T(\theta) = t(X, r_0(X_r), \theta)$ is continuously distributed with compact support I_T satisfying $I_T^* \subset$

int(I_T) *with* I_T^* *compact. The corresponding density function* $f_T(\cdot,\theta)$ *is continuously differentiable for every* $\theta \in \Theta$*, and* $\inf_{\theta\in\Theta, x\in I_T^*} f_T(x,\theta) > 0$.

(iii) For every $\theta \in \Theta$*, the functions* $m_0(\cdot,\theta)$ *and* $t(\cdot,\theta)$ *are* $(p+1)$*-times continuously differentiable on their respective domains.*

(iv) For a constant $C > 0$ *it holds that* $E[\exp(l|Y|)] \le C$ *for* $l > 0$ *small enough.*

(v) The function $\mathcal{K}$ *is twice continuously differentiable and satisfies the following conditions:* $\int \mathcal{K}(u)du = 1$*,* $\int u\mathcal{K}(u)du = 0$ *and* $\int |u^2\mathcal{K}(u)|du < \infty$*, and* $\mathcal{K}(u) = 0$ *for values of* u *not contained in some compact interval, say* $[-1, 1]$.

(vi) The bandwidth $h = (h_1, \ldots, h_{d_T})$ *satisfies* $h_j \sim n^{-\eta_j}$ *for all* $j = 1, \ldots, d_T$*, and* $(1 - \eta_+)/2 > \eta_{\max}$.

Most restrictions imposed in Assumption 1 are standard for nonparametric kernel-type estimators of nuisance functions in semiparametric models. Part (i) is not necessary and could be relaxed to allow for certain forms of temporal dependence. Part (ii) introduces a "fixed trimming" procedure, ensuring a stable estimate $\widehat{m}(\cdot,\theta)$ at the points of evaluation. The differentiability conditions in (iii) are used to control the magnitude of bias terms. Assuming subexponential tails of ε conditional on $T(\theta)$ in part (iv) is necessary to apply certain results from empirical process theory in our proofs. Part (v) describes a standard kernel function with compact support. Finally, the restrictions on the bandwidth in (vi) imply that those bias terms are dominated by certain stochastic terms.

Assumption 2 (Accuracy)**.** *We assume the following properties of the estimator* $\widehat{r}$*:*

(i) $\sup_s |\widehat{r}_j(s) - r_{0,j}(s)| = o_P(n^{-\delta_j^*})$ *for some* $\delta_j^* > 1/4$ *and all* $j = 1, \ldots, d_r$*, and*

(ii) $\sup_{\theta,x} |T_j(x,\theta,\widehat{r}) - T_j(x,\theta,r_0)| = o_P(n^{-\delta_j})$ *for some* $\delta_j > \eta_j$ *and all* $j = 1, \ldots, d_t$*,*

where in both cases the subscript j *denotes the* j*-th component of the respective object.*

Assumption 2 imposes restrictions on the accuracy of the first-step estimator $\hat{r}$: part (i) is needed for condition (N4) of Theorem 1 to hold, whereas part (ii) ensures that the difference between the respective components of $\widehat{T}(\theta)$ and $T(\theta)$ tend to zero in probability at a rate as least as fast as the corresponding bandwidth in the second stage of the estimation procedure, uniformly in θ. Such conditions can be verified for a wide range of nonparametric estimators (e.g. Masry (1996), Newey (1997)), and they trivially hold for regular parametric estimators.

Assumption 3 (Complexity). *For every $j = 1, \ldots, d_T$, there exist a sequence of sets of functions $\mathcal{T}_{n,j}$ such that*

(i) $\Pr(T_j(\cdot, \widehat{r}) \in \mathcal{T}_{n,j}) \to 1$ *as* $n \to \infty$.

(ii) For a constant $C_T > 0$ and a function r_n with $\|T_j(x, \theta, r_n) - T_j(x, \theta, r_0)\|_\infty = o_P(n^{-\delta_j})$, the set $\mathcal{T}^*_{n,j} = \mathcal{T}_{n,j} \cap \{T_j(\cdot, r) : \|T_j(x, \theta, r) - T_j(x, \theta, r_n)\|_\infty \leq n^{-\delta_j}\}$ can be covered by at most $C_T \exp(\lambda^{-\alpha_j} n^{\xi_j})$ balls with $\|\cdot\|_\infty$-radius λ for all $\lambda \leq n^{-\delta_j}$, where $0 < \alpha_j \leq 2$, $\xi_j \in \mathbb{R}$ and $\|\cdot\|_\infty$ denotes the supremum norm in $x \in \mathcal{X}$ and $\theta \in \Theta$.

Assumption 3 restricts the complexity of the function space in which the mapping $(x, \theta) \mapsto T(x, \theta, \widehat{r})$ takes its values by imposing constraints on the cardinality of the covering sets. Since we have that $T(x, \theta, r) = t(x, r(x_r), \theta)$ for some known function t which, by Assumption 1(iii), is continuously differentiable with respect to its second component, the condition imposes implicit restrictions on the complexity of the first-stage estimator $\widehat{r}$. Indeed, we could equivalently state a restriction similar to Assumption 3 on the set $\mathcal{R}^*_n = \{r \in \mathcal{R} : T_j(\cdot, r) \in \mathcal{T}^*_{n,j}$ for all $j = 1, \ldots, d_T\}$.

Restrictions on covering numbers are a common requirement in the literature on empirical processes, that is typically fulfilled under suitable smoothness assumptions. Suppose for example that $\mathcal{R}^*_n$ is the set of smooth functions defined on the compact set $I_R \subset \mathbb{R}^{d_{X_r}}$, whose partial derivatives up to order k exist and are uniformly bounded by some multiple of $n^{\xi^*_j}$ for some $\xi^*_j \geq 0$, and that $|T_j(x, r(x_r), \theta) - T_j(x, r(x_r), \theta^*)| \leq C\|\theta - \theta^*\|$ for every θ, θ^* and every value of x and r. Then the set $\mathcal{T}_{n,j}$ satisfies Assumption 3(ii) with $\alpha_j = d_{X_r}/k$ and $\xi_j = \xi^*_j \alpha_j$ (Van der Vaart and Wellner, 1996, Corollary 2.7.2). The same entropy bound applies if $\mathcal{R}^*_n$ consists in the sum of one fixed function and a smooth function from a respective smoothness class. This extension is useful if one chooses the fixed function as equal to the sum of r_0 and the bias of $\widehat{r}$. Thus it is not necessary that the bias term is a smooth function. In a setting where r_0 is estimated by parametric or semiparametric methods, substantially smaller values can be established for the constants α_j and ξ_j. See e.g. van de Geer (2009) for further discussion and examples.

Assumption 4 (Continuity)**.** *We assume that the elements of* $\mathcal{R}_n^* = \{r \in \mathcal{R} : T_j(\cdot, r) \in \mathcal{T}_{n,j}^*$ *for all* $j = 1, \dots, d_T\}$ *satisfy the following properties:*

(i) For all $r \in \mathcal{R}_n^*$ *and* $\theta \in \Theta$ *the function* $\tau^B(t, \theta, r) = \mathbb{E}(\rho(X, \theta)|T(r) = t)$ *with* $\rho(X, \theta) = \mathbb{E}(Y|X) - \mathbb{E}(Y|T(\theta))$ *is* $p + 1$ *times differentiable with respect to its first argument, and the derivatives are uniformly bounded in absolute value.*

(ii) For a constant $C_B^* > 0$ *and for* $r_1, r_2 \in \mathcal{R}_n^*, \theta \in \Theta$ *it holds that*

$$\|\tau^B(T(r_1), \theta, r_1) - \tau^B(T(r_2), \theta, r_2)\| \leq C_B^* \|r_1 - r_2\|_\infty \text{ a.s.}$$

(iii) For a constant $C_B^* > 0$ *and all* $r_1, r_2 \in \mathcal{R}_n^*, \theta \in \Theta$ *and* $t \in I_T^*$ *it holds that*

$$\Big|\mathbb{E}\left[(T(\theta, r_1) - t)^u h^{-u} K_h(T(\theta, r_1) - t)\right] - \mathbb{E}\left[(T(\theta, r_2) - t)^u h^{-u} K_h(T(\theta, r_2) - t)\right]\Big| \leq C_B \|r_1 - r_2\|_\infty$$

for $0 \leq u_+ \leq p$.

Assumption 4(i)–(ii) are technical conditions that ensure that the conditional expectation of the "index bias" $\rho(X, \theta)$ satisfies certain smoothness restrictions. In certain applications, we have that $\rho(X, \theta) = 0$ with probability 1, and thus these conditions trivially hold. Assumption 4(iii) is a further smoothness condition. If the random vector $r(X_r)$ is continuously distributed, this condition holds if $\|f_1 - f_2\|_\infty \leq C_B \|r_1 - r_2\|_\infty$ for all $r_1, r_2 \in \mathcal{R}_n^*$, where f_j denotes the density function of $r_j(X_r)$ for $j = 1, 2$. See Escanciano, Jacho-Chávez, and Lewbel (2011, Assumption 10) for a similar restriction on the densities of the generated covariates.

3.2. Stochastic Expansions of the Nonparametric Component. Using the assumptions outlined above, we can now derive a sharp stochastic approximation of the nonparametric estimator $\widehat{m}$. To state the result, we denote the unit vector $(1, 0, \dots, 0)^\top$ in $\mathbb{R}^{p+1}$ by e_1, and write $w_i(t, \theta, r) = (1, (T_i(r, \theta) - t)/h, ..., (T_i(r, \theta) - t)^p/h^p)^\top$ and $N_h(x, \theta) = \mathbb{E}(w_i(t, \theta, r) w_i(t, \theta, r)^\top K_h(T_i(r, \theta) - t))$. Recalling that $\rho(X, \theta) = \mathbb{E}(Y|X) - \mathbb{E}(Y|T(\theta))$, we then define the approximating function $\widehat{m}_\Delta$ by

$$\widehat{m}_\Delta(t, \theta) = \tilde{m}(t, \theta) + \varphi_n^A(t, \theta, \widehat{r}) + \varphi_n^B(t, \theta, \widehat{r}), \tag{3.2}$$

where

$$\varphi_n^A(t,\theta,r) = -m_0'(t,\theta)e_1^\top N_h(x,\theta)^{-1}\mathbb{E}(K_h(T_i(\theta)-t)w_i(x,\theta)(T_i(r,\theta)-T_i(\theta)))$$

in case of local linear regression with $p=1$ (a general, notationally much more involved definition for higher order local polynomials is given in (A.2) in Appendix A), and

$$\varphi_n^B(t,\theta,r) = e_1^\top N_h(x,\theta)^{-1}\mathbb{E}(K_h'(T_i(\theta)-t)^\top w_i(x,\theta)(T_i(r,\theta)-T_i(\theta))\rho(X,\theta))$$

for any $r \in \mathcal{R}_n^*$. Here we use the notation $K_h'(v) = (\mathcal{K}_{h,j}'(v) : j = 1,...,d_T)^\top$ with elements $\mathcal{K}_{h,j}'(v) = \mathcal{K}'(v_j/h_j)/h_j^2 \prod_{j^*\neq j}\mathcal{K}(v_{j^*}/h_{j^*})/h_{j^*}$. Our main result concerns the accuracy when using $\widehat{m}_\Delta$ as an approximation of $\widehat{m}$.

Theorem 2. *Suppose that Assumption 1–4 hold. Then for any $\theta \in \Theta$, it is*

$$\int(\widehat{m}(t,\theta)-\widehat{m}_\Delta(t,\theta))\omega(x)dx = o_p(n^{-\kappa^*}) \tag{3.3}$$

for some weight function $\omega : \mathbb{R}^d \to \mathbb{R}$ whose partial derivatives of order one are uniformly absolutely bounded, and that satisfies $\omega(x) = 0$ for all $x \notin I_T^*$, and $\kappa^* = \min\{\kappa_1^*,\ldots,\kappa_4^*\}$ with

$$\kappa_1^* = \frac{1}{2} + (1-\frac{\alpha_{max}}{2})\delta_{min} - \frac{(\alpha\eta+\xi)_{max}}{2},\ \kappa_2^* < (p+1)\eta_{min} + (\delta-\eta)_{min},$$
$$\kappa_3^* < (2-\frac{\alpha_{max}}{2})\delta_{min} + \frac{1}{2}(1-\eta_+) - \frac{(\alpha\eta+\xi)_{max}}{2},\ \kappa_4^* < 2\delta_{min}.$$

The Theorem provides a sharp bound on weighted averages the the approximation error $\widehat{m}(t,\theta)-\widehat{m}_\Delta(t,\theta)$. We focus on this class of distance measures because they are particularly suitable to verify conditions of the type (N6) in Theorem 1. Bounds on the supremum norm of the approximation error, as studied Mammen, Rothe, and Schienle (2011), typically vanish at a rate slower than $n^{-1/2}$, and are thus not useful to establish the "asymptotic normality" condition. They can however, with some adaptaion, be employed to verify the "uniform consistency" condition (N4), as explained below.

The function $\widehat{m}_\Delta$ consists of two components: the term $\tilde{m}(\cdot,\theta)$ is the oracle estimator of $m_0(\cdot,\theta)$ introduced above, whereas $\varphi_n^A(t,\theta,\widehat{r})+\varphi_n^B(t,\theta,\widehat{r})$ is an adjustment term that captures the additional uncertainty due to the presence of generated covariates. Note that the generated covariates enter the expansion only through *smoothed* versions of the estimation error $T(\theta, \hat{r}) - T(\theta, r_0)$. Since this additional smoothing typically improves the rate of convergence of the stochastic part of the first-step estimator (although it does not improve the order of the bias component), we generally expect the adjustment term to have a faster rate of convergence. Hence the dimensionality of the generation step should play a less pronounced role in this context.

4. Application to Semiparametric Estimation

In this section, we show how to verify conditions of the type (N4) and (N6) in Theorem 1. We also derive a general formula for the asymptotic variance of the estimator $\hat{\theta}$. Throughout the section, we assume that the smoothness conditions (N2)–(N3) on the criterion function Q hold.

4.1. Verifying "Uniform Consistency". To verify the "Uniform Consistency" condition (N4), we use a variation of an earlier result in Mammen, Rothe, and Schienle (2011) to derive the uniform rate of consistency of the estimator $\widehat{m}(t, \theta)$.

Theorem 3 (Uniform Consistency)**.** *Suppose Assumption 1–3 and 4(i)–(ii) hold. Then*

$$\sup_{t \in I_T^*, \theta \in \Theta} |\widehat{m}(t, \theta) - m_0(t, \theta)| = O_p\left(n^{-(p+1)\eta_{min}} + \sqrt{\log(n) n^{-(1-\eta_+)}} + n^{-\delta_{min}} + n^{-\kappa} \right),$$

where $\kappa = \min\{\kappa_1, ..., \kappa_3\}$ *with*

$$\kappa_1 < \frac{1}{2}(1 - \eta_+) + (\delta - \eta)_{min} - \frac{1}{2}(\delta\alpha + \xi)_{max}, \ \kappa_2 < (p+1)\eta_{min} + (\delta - \eta)_{min},$$
$$\kappa_3 < \delta_{min} + (\delta - \eta)_{min}.$$

The first two terms in the error bound on the right hand side follow from a standard uniform consistency result of the oracle estimator $\widetilde{m}$ (Masry, 1996), whereas the remaining two terms are due to the presence of generated covariates. In order for condition (N4) to hold, these terms have to be of smaller order than $n^{-1/4}$. For the oracle part, this can easily be achieved by choosing an appropriate bandwidth under sufficient smoothness conditions. For the remaining terms, Assumption 2(i) and Assumption 1(iii) jointly imply that $\delta_{min} > 1/4$. It then follows from simple calculations that $O_p\left(n^{-\delta_{min}} + n^{-\kappa}\right) = o_p(n^{-1/4})$ under appropriate restrictions on the sets $\mathcal{T}_{n,j}$.[3]

[3]Note that when studying the "asymptotic normality" condition (N6) in the next subsection, we will

4.2. Verifying "Asymptotic Normality". Given a specific estimator $\widehat{r}$ of r_0, the expansion $\widehat{m}_\Delta(t,\theta)$ in (3.2) can usually be calculated more explicitly, and can then be used to verify (N6). To illustrate this idea in a general setting, suppose that the estimator used to generate the covariates satisfies the following asymptotically linear representation, which can be shown to be satisfied for a wide range nonparametric, semiparametric, and fully parametric estimation procedures (we discuss two representative examples below).

Assumption 5 (Linear Representation)**.** *The estimator $\widehat{r}$ of r_0 satisfies*

$$\widehat{r}(s) - r_0(s) = \frac{1}{n}\sum_{i=1}^{n} \varphi_{ni}^{\widehat{r}}(s) + R_n(s) \tag{4.1}$$

with $\varphi_{ni}^{\widehat{r}}(s) = \mathcal{H}_n(S_i, s)\nu(W_i)$ for some $S_i \subset W_i$ and $\sup_{s\in I_R^*} |R_n(s)| = o_p(n^{-1/2})$. The term $\nu(W_i)$ satisfies $\mathbb{E}(\nu(W_i)|S_i) = 0$ and $\mathbb{E}(\nu(W_i)\nu(W_i)^\top) < \infty$, and $\mathcal{H}_n$ is a weighting function satisfying $\mathbb{E}(\|\mathcal{H}_n(S_i, S_j)\|^2) = o(n)$ for $i \neq j$.

To see how this additional structure can be utilized for our purposes, recall that it follows from elementary rules for pathwise derivatives that

$$Q_0^\xi[\hat{\xi} - \xi_0] = Q^m(\theta_0, \xi_0)[\hat{m} - m_0] + Q^r(\theta_0, \xi_0)[\hat{r} - r_0],$$

where for any (θ, r) the functional $Q^m(\theta, \xi)[\bar{m}]$ is the pathwise derivative of $Q(\theta, (m, r))$ at m in the direction $\bar{m}$, and similarly for Q^r. In most applications, m and r are square integrable functions of random vectors Z_m and Z_r, respectively, and it follows from the Riesz representation theorem that there exists unique square integrable functions λ_m and λ_u such that

$$Q^m(\theta_0, \xi_0)[\hat{m} - m_0] = \int \lambda_m(z)(\hat{m}(z) - m_0(z))dF_{Z_m}(z), \tag{4.2}$$

$$Q^r(\theta_0, \xi_0)[\hat{r} - r_0] = \int \lambda_r(z)(\hat{r}(z) - r_0(z))dF_{Z_r}(z). \tag{4.3}$$

See e.g. Newey (1994). The form of λ_m and λ_r depends on the particular application. For example, if the criterion function $Q(\theta, \xi) = \mathbb{E}(q(Z, \theta, m, r))$ is such that the term

introduce some additional structure on the estimator $\widehat{r}$ of r_0 in Assumption 5. Using this additional structure, it would be possible to derive better rates than the one given in Theorem 3. See the remark at the end of the proof of Theorem 3 in Appendix A for details.

$q(Z, \theta, m, r)$ only depends on the functions m and r smoothly through their value when evaluated at some random vectors Z_m and Z_r, respectively, we have that

$$\lambda_m(z_m) = \mathbb{E}(\partial q(Z, \theta, m_0, r_0)/\partial m_0(Z_m, \theta_0)|Z_m = z_m)$$
$$\lambda_r(z_r) = \mathbb{E}(\partial q(Z, \theta, m_0, r_0)/\partial r_0(Z_r)|Z_r = z_r).$$

All econometric applications we consider in Section 5 below exhibit this structure.

When λ_m and λ_r are sufficiently smooth, one can use Assumption 5 together with the representation in (3.2) to show that there exist fixed functions ψ_j with $\mathbb{E}(\psi_j(Z)) = 0$ and $\mathbb{E}(\psi_j(Z)\psi_j(Z)^\top) < \infty$ for $j = 1, 2, 3$ such that

$$\int \lambda_m(z)\tilde{m}(z, \theta_0)dF_{Z_m}(z) = \frac{1}{n}\sum_{i=1}^n \psi_1(Z_i) + o_p(n^{-1/2})$$
$$\int \lambda_m(z)\left(\frac{1}{n}\sum_{i=1}^n \varphi_{ni}^A(z, \theta_0, \hat{r}) + \varphi_n^B(z, \theta_0, \hat{r})\right) dF_{Z_m}(z) = \frac{1}{n}\sum_{i=1}^n \psi_2(Z_i) + o_p(n^{-1/2}),$$
$$\int \lambda_r(z)\frac{1}{n}\sum_{i=1}^n \varphi_{ni}^{\hat{r}}(z)dF_{Z_r}(z) = \frac{1}{n}\sum_{i=1}^n \psi_3(Z_i) + o_p(n^{-1/2}).$$

Moreover, the properties of the remainder term $R_n(t) = \widehat{m}(t, \theta_0) - \widehat{m}_\Delta(t, \theta_0)$ established in Theorem 2 ensure, under suitable regularity conditions, that

$$\int \lambda_m(z)R_n(z)dF_{Z_m}(z) = o_p(n^{-1/2}).$$

If we now put $\psi_0(Z_i) = q(Z_i, \theta_0, \xi_0)$ and $\psi(z) = \sum_{j=0}^3 \psi_j(z)$, the above statements imply that

$$\sqrt{n}(Q_n(\theta_0, \xi_0) + Q_0^\xi[\hat{\xi} - \xi_0]) = \frac{1}{\sqrt{n}}\sum_{i=1}^n \psi(W_i) + o_p(1) \xrightarrow{d} N(0, \mathbb{E}(\psi(Z)\psi(Z)^\top)) \quad (4.4)$$

by the Central Limit Theorem, and thus condition (N6) holds with $V = \mathbb{E}(\psi(Z)\psi(Z)^\top)$. The following Corollary formalizes this argument, and provides a general formula to compute the variance matrix V.

Corollary 1 (Normality). *Suppose Assumption 1– 5 holds with $p+1 > d_T$,*

$$\frac{(\alpha\eta + \xi)_{max}}{2} < \min\{(1 - \frac{\alpha_{max}}{2})\delta_{min}, (2 - \frac{\alpha_{max}}{2})\delta_{min} + \frac{1}{2}(1 - \eta_+)\}, \quad (4.5)$$

the criterion function satisfies (4.2)– (4.3) *with $\lambda_m(\cdot)$ and $\lambda_r(\cdot)$ being $(p+1)$-times continuously differentiable, and $1/2(p+1) < \eta_j < 1/2d_t$ for $j = 1, \ldots, d_T$. Then equation* (4.4) *holds with*

$$\psi_1(Z_i) = \varepsilon_i \lambda_m(T_i) f_{Z_m}(T_i) f_T(T_i)^{-1}$$
$$\psi_2(Z_i) = -\nu(W_i)\mathbb{E}(\lambda_m^*(X_r)\mathcal{H}_n(S_i, X_r)|S_i)$$
$$\psi_3(Z_i) = \nu(W_i)\mathbb{E}(\lambda_r(X_r)\mathcal{H}_n(S_i, X_r)|S_i),$$

where

$$\lambda_m^*(x_r) = \mathbb{E}(T^{(r)}(X)(\rho(X)G'(T) + m_0'(T)G(T))|X_r = x_r)$$

and $G(t) = \lambda_m(t) f_{Z_m}(t) f_T(t)^{-1}$ *and* $G'(t) = \partial_t G(t)$ *and* $T^{(r)}(x) = \partial T(x, \theta_0, r_0)/\partial r_0(x_r)$.

Restriction (4.5) involves a tradeoff between the complexity of the first and second estimation step for the nonparametric component: It can be shown to be satisfied when r_0 is "sufficiently regular" (i.e. the α_j and ξ_j are small) and $m_0(\cdot, \theta)$ is "sufficiently smooth" (i.e. p is large and thus the η_j can be chosen small). Exact conditions are difficult to give in general, but are easy to check for a specific application, where specific values for the α_j and ξ_j are available. See the discussion after Assumption 3 above for an example.

Assumption 5 is similar to conditions used e.g. in Rothe (2009) or Ichimura and Lee (2010). We now give two examples for which it is satisfied: the case where r_0 is a conditional expectation function estimated by nonparametric regression, and the case where $r_0(x_r) = \bar{r}(x_r, \vartheta_0)$ is a function known up to a finite dimensional parameter ϑ_0, for which there exists a regular asymptotically linear estimator. These are arguably the most important cases from an applied point of view.

Example 1 (Nonparametric Regression)**.** Suppose that W is partitioned as $W = (D, S)$, and we have that $D = r_0(S) + \zeta$ with $\mathbb{E}(\zeta|S) = 0$. Consider a kernel-based nonparametric regression estimator $\hat{r}$ of r_0, such as the Nadaraya-Watson or a local polynomial estimator. Then one can show that Assumption 5 holds under suitable smoothness conditions with $\nu(W_i) = \zeta_i$ and $\mathcal{H}_n(S_i, s) = f_S(s)^{-1} L_g(S_i - s)$, where L is a kernel function and g is a bandwidth that tends to zero at an appropriate rate. We then find that

$$\psi_2(Z_i) = -\zeta_i \lambda_m^*(S_i) \frac{f_{X_r}(S_i)}{f_S(S_i)} \quad \text{and} \quad \psi_3(Z_i) = \zeta_i \lambda_r(S_i) \frac{f_{Z_r}(S_i)}{f_S(S_i)}.$$

The form of $\psi_0(\cdot)$ and $\psi_1(\cdot)$ remain unchanged. □

Example 2 (Nonlinear Parametric Estimation)**.** Assume that $r_0(x_r) = \bar{r}(x_r, \vartheta_0)$ is a parametrically specified function (not necessarily a conditional expectation) known up to the finite dimensional parameter ϑ_0. Suppose there exists an estimator $\hat{\vartheta}$ of ϑ_0 that satisfies

$$\widehat{\vartheta} - \vartheta_0 = \frac{1}{n}\sum_{i=1}^{n} \varphi^{\hat{\vartheta}}(W_i) + o_p(n^{-1/2}),$$

where $\mathbb{E}(\varphi^{\hat{\vartheta}}(W)) = 0$, $\mathbb{E}(\varphi^{\hat{\vartheta}}(W)\varphi^{\hat{\vartheta}}(W)^\top) < \infty$, and that $r(x_r, \mu)$ is continuously differentiable in its second argument. Then Assumption 5 is satisfied with $\nu(W_i) = \varphi^{\hat{\vartheta}}(W_i)$ and $\mathcal{H}_n(S_i, x_r) = \partial_\vartheta r(x_r, \vartheta_0)$, and thus

$$\begin{aligned}
\psi_2(Z_i) &= -\nu(W_i)\mathbb{E}(T^r(X)\partial_\vartheta r(X_r, \vartheta_0)(\rho(X)g(T) + \lambda_m(T)m_0'(T)f_{Z_m}(T)f_T(T)^{-1})) \\
\psi_3(Z_i) &= \nu(W_i)\mathbb{E}(\lambda_r(X_r)\partial_\vartheta r(X_r, \vartheta_0)).
\end{aligned}$$

In case that $r_0(x_r) = \bar{r}(x_r, \vartheta_0)$ is a regression function estimated by nonlinear least squares, we have that $\nu(W_i) = \mathbb{E}(\partial_\vartheta r(X_r, \vartheta_0)\partial_\vartheta r(X_r, \vartheta_0)^\top)^{-1}\partial_\vartheta r(X_{r,i}, \vartheta_0)(D_i - r_0(S_i))$, under the usual conditions. □

4.3. The Asymptotic Variance. The argument in the previous subsection conveys some important intuition for the form of the asymptotic variance of $\hat{\theta}$. Recall that under the conditions of Theorem 1 this variance is given by

$$\Omega = (Q_0^{\theta\top} A Q_0^\theta)^{-1} Q_0^{\theta\top} A V A Q_0^\theta (Q_0^{\theta\top} A Q_0^\theta)^{-1}$$

with $V = \mathbb{E}(\psi(Z)\psi(Z)^\top)$ and $\psi(z) = \sum_{j=0}^{3} \psi_j(z)$. In contrast, the asymptotic variance of the oracle estimator $\tilde{\theta}$ can be shown to be

$$\tilde{\Omega} = (Q_0^{\theta\top} A Q_0^\theta)^{-1} Q_0^{\theta\top} A \tilde{V} A Q_0^\theta (Q_0^{\theta\top} A Q_0^\theta)^{-1}$$

with $\tilde{V} = \mathbb{E}((\psi_0(Z) + \psi_1(Z))(\psi_0(Z) + \psi_1(Z))^\top)$, by simply setting $\widehat{r} = r_0$. The presence of generated covariates thus affects the asymptotic variance only through the additional summands $\psi_2(Z)$ and $\psi_3(Z)$ used to calculate V, as the weight matrix A is chosen by the econometrician and Q_0^θ is simply a population quantity. In particular, the term $\psi_2(Z)$ captures the additional uncertainty due to using generated covariates when *estimating* the function m_0, whereas the term $\psi_3(Z)$ accounts for *directly using* the generated covariates in other parts of the model, e.g. as a point of evaluation of an estimated function. A simple condition for the presence of generated covariates to be asymptotically negligible, i.e. that $\Omega = \tilde{\Omega}$, is then of course that $\psi_2(Z) = -\psi_3(Z)$ with probability one. This finding generalizes recent results in Hahn and Ridder (2011), who were the first to derive the influence function for a class of semiparametric estimators with generated covariates.

Remark 1 (Asymptotic Variance for a Special Case)**.** Hahn and Ridder (2011) consider a special case of our setup where $T(X, \theta, r) = (X_1, r(X_r))$ and the criterion function of the form $Q_n(\theta, m, r) = n^{-1} \sum_{i=1}^n q(Z_i, \theta, m, r)$ with $q(Z, \theta, m, r) = s(m((X_1, r(X_r)))) - \theta$ for some known function s. In this setting, one can give intuitive conditions under which the presence of generated covariates is asymptotically negligible. Suppose for example that r_0 is a nonparametric regression function satisfying $D = r_0(X_r) + \zeta$ with $\mathbb{E}(\zeta|X_r) = 0$. Applying Corollary 1 as in Example 1 above, we find that in this setting the asymptotic variance of the estimator is given by[4]

$$\Omega = \mathbb{E}((\Psi_1 + \Psi_2)(\Psi_1 + \Psi_2)^\top)$$

where, writing $T = (X_1, r_0(X_r))$,

$$\begin{aligned}\Psi_1 &= s(m_0(T)) - \theta + s'(m_0(T))\varepsilon, \\ \Psi_2 &= -\zeta\mathbb{E}(s''(m_0(T))m_0'(T)T^{(r)}(X)(Y - E(Y|T))|X_r).\end{aligned}$$

Here the term $\Psi_2 = \psi_2(Z) + \psi_3(Z)$ accounts for the estimation error from using an estimate of r_0 instead of the actual function, and is easily seen to be equal to zero if either $s(\cdot)$ is a linear function or $\mathbb{E}(Y|X) = E(Y|T)$.

Remark 2 (Validity of the Bootstrap)**.** In some applications, the asymptotic variance matrix V could be difficult to estimate since it depends on the nonparametrically estimated components of the model in a potentially nontrivial fashion. In such cases, resampling techniques like the ordinary nonparametric bootstrap can be useful to compute confidence regions for the parameters of interest. Our results can be used to establish the validity of such an approach. Consider for example the setting in Chen, Linton, and Van Keilegom (2003), where $Q_n(\theta, \xi) = n^{-1} \sum_{i=1}^n q(Z_i, \theta, m(Z_{m,i}, \theta), r(Z_{r,i}))$

[4] The same formula is also derived by Hahn and Ridder (2011) in their Theorem 3.

and $Q(\theta, \xi) = \mathbb{E}(q(Z, \theta, m(Z_m, \theta), r(Z_r)))$. Let $(Z_1^*, \ldots, Z_n^*)$ be be drawn with replacement from the original sample $(Z_1, \ldots, Z_n)$, let $\widehat{\xi}^*$ be the same estimator as $\widehat{\xi}$ but based on the bootstrap data, and put $Q_n^*(\theta, \xi) = n^{-1} \sum_{i=1}^n q(Z_i^*, \theta, m(Z_{m,i}^*, \theta), r(Z_{r,i}^*))$. Next, define the bootstrap estimator $\widehat{\theta}^*$ as any sequence that minimizes a GMM-type criterion function based on a recentered moment condition:

$$\|Q_n^*(\hat{\theta}^*, \hat{\xi}^*) - Q_n(\hat{\theta}, \hat{\xi})\| = \inf_{\theta \in \Theta} \|Q_n^*(\hat{\theta}, \hat{\xi}^*) - Q_n(\hat{\theta}, \hat{\xi})\| + o_{p^*}(1/\sqrt{n}).$$

Chen, Linton, and Van Keilegom (2003) give sufficient condition under which the distribution of $\sqrt{n}(\widehat{\theta}^* - \widehat{\theta})$ converges in distribution to $N(0, V)$ under the probability measure implied by the bootstrap. Following the discussion after their Theorem B, these conditions can be verified by the same arguments we used to establish (N4) and (N6) above, and are thus immediate for a wide range of applications.

4.4. Relationship to Recent Literature. The results in our paper are closely related to recent findings in Hahn and Ridder (2011) and Escanciano, Jacho-Chávez, and Lewbel (2011). In this subsection, we discuss the differences in detail.

Remark 3 (Relationship to Hahn and Ridder (2011))**.** In an important related paper, Hahn and Ridder (2011) study the form of the influence function of semiparametric linear, just-identified GMM-type estimators in the presence of generated covariates, using pathwise derivatives as in Newey (1994). They do not consider a particular estimation procedure, but assume that the estimator satisfies the asymptotically linear representation

$$\hat{\theta} = \theta_0 + n^{-1} \sum_{i=1}^n \psi(Z_i) + o_p(n^{-1/2}) \tag{4.6}$$

with $\mathbb{E}(\psi(Z)) = 0$ and $\mathbb{E}(\psi(Z)\psi(Z)^\top) < \infty$. Under this assumption, they derive a formula for the function ψ for their class of semiparametric models. However, they do not study conditions that ensure the validity of the representation (4.6) in the first place, which is by no means self evident. Their analysis does thus not imply that a particular estimator is root-n consistent and asymptotically normal.

Our paper complements and extends the work of Hahn and Ridder (2011) in several important ways. First, we consider a strictly larger class of estimators, allowing e.g. for profiled optimization estimators with non-smooth criterion functions. Second, and more importantly, using our stochastic expansions we provide explicit conditions for root-n consistency and asymptotic normality for estimators contained in this larger class.[5] We also derive a general formula for the asymptotic variance of our estimators, and show how to establish validity of the bootstrap, which is important for many empirical applications.

Remark 4 (Relationship to Escanciano, Jacho-Chávez, and Lewbel (2011))**.** In another closely related paper, Escanciano, Jacho-Chávez, and Lewbel (2011) derive stochastic expansions for sample means of weighted semiparametric regression residuals. Their results can be used to study the asymptotic properties of estimators in certain semiparametric "index models" with generated covariates, such as e.g. those with (in our notation) a criterion function of the form $Q_n(\theta, m, r) = n^{-1} \sum_{i=1}^n (Y_i - m(T(X_i, \theta, r), \theta))s(X_i)$, where $s(X)$ is some weighting term.[6] Such models are contained in the general class we consider in this paper. Escanciano, Jacho-Chávez, and Lewbel (2011) use stochastic equicontinuity arguments to control the impact of generated regressors on the final estimator, which rely on a certain functional Lipschitz condition (their Assumption 7) that seems difficult to verify in practice. In contrast, our results are derived using more direct bounds to control the impact of generated covariates, and can thus be applied without verifying such a condition.

A further important difference is that Escanciano, Jacho-Chávez, and Lewbel (2011) assume that $\mathbb{E}(Y|T) = \mathbb{E}(Y|X)$ in their models, i.e. that the index T is a sufficient statistic for the random vector X. As described above, this condition is often not satisfied in applications, such as e.g. the estimation of average treatment effects we study in Section 5.1. Our results do not require such an assumption. To illustrate the implications of this condition, consider the example mentioned above where $Q_n(\theta, m, r) = n^{-1} \sum_{i=1}^n (Y_i - m(T(X_i, \theta, r), \theta))s(X_i)$, and suppose again that the function r_0 is a non-

[5] Due to the flexibility of our stochastic expansions, we conjecture that it should also be possible to extend our analysis to semiparametric estimators that are asymptotically normal but do not satisfy an asymptotic linearity condition, as studied e.g. by Cattaneo, Crump, and Jansson (2011).

[6] The results in Escanciano, Jacho-Chávez, and Lewbel (2011) are substantially more general, as they allow for estimated weights, the presence of vanishing trimming terms, and data-dependent choices of the bandwidth. These features make their results very useful even for model not involving generated covariates.

parametric regression function that satisfies $D = r_0(X_r) + \zeta$ with $\mathbb{E}(\zeta|X_r) = 0$. Applying Corollary 1 as in Example 1, we find that the asymptotic variance of the estimator in this setting is equal to

$$\Omega = Q_0^{\theta-1}\mathbb{E}((\Psi_1 + \Psi_2 + \Psi_3)(\Psi_1 + \Psi_2 + \Psi_3)^\top)Q_0^{\theta-1},$$

where, writing $u(t) = \mathbb{E}(s(X)|T = t)$,

$$\begin{aligned}
\Psi_1 &= \varepsilon(s(X) - \mathbb{E}(s(X)|T)) \\
\Psi_2 &= -\zeta\mathbb{E}((s(X) - \mathbb{E}(s(X)|T))m_0'(T)T^{(r)}(X)|X_r) \\
\Psi_3 &= \zeta\mathbb{E}(u'(T)T^{(r)}(X)(\mathbb{E}(Y|X) - E(Y|T))|X_r).
\end{aligned}$$

The terms Ψ_2 and Ψ_3 account for the estimation error from using an estimate of r_0 instead of the actual function. The expansion in Escanciano, Jacho-Chávez, and Lewbel (2011) can be used to obtain a similar result under their stronger conditions; see their Corollary 2.1. Since they impose that $\mathbb{E}(Y|X) = E(Y|T)$ the term Ψ_3 is equal to zero in this case.

5. Econometric Applications

Semiparametric estimation problems with generated covariates occur in various fields of econometrics. In this subsection, we discuss two applications in greater detail: estimation of average treatment effects via regression on the propensity score, and estimation of production functions in the presence of serially correlated technology shocks. To save space, we only sketch the construction of estimators, and refer to Appendix B for details and regularity conditions.

5.1. Regression on the Propensity Score. Consider the potential outcomes framework, which is commonly used in the literature on program evaluation (Imbens, 2004): Let Y_1 and Y_0 be the potential outcomes with and without program participation, respectively, $D \in \{0, 1\}$ an indicator of program participation, $Y = Y_1 D + Y_0(1 - D)$ be the observed outcome, X a vector of exogenous covariates, and let $\Pi(x) = \Pr(D = 1|X = x)$ be the propensity score. A typical object of interest in this context is the average treatment effect (ATE), defined as

$$\theta_0 = \mathbb{E}(Y_1 - Y_0).$$

Since selection into the program may be nonrandom, this object cannot be obtained by simply comparing the average outcomes of treated and untreated individuals. However, when selection depends on observable covariates X only, biases due to nonrandom selection into the program can be removed by conditioning on the propensity score (Rosenbaum and Rubin, 1983). That is, the condition that $Y_1, Y_0 \perp D|X$ implies that $Y_1, Y_0 \perp D|\Pi(X)$. Moreover, writing $\nu_d(\pi) = \mathbb{E}(Y|D = d, \Pi(X) = \pi)$, we have that $\nu_d(\pi) = \mathbb{E}(Y_d|\Pi(X) = \pi)$, and thus by the law of iterated expectations, the ATE is identified through the relationship

$$\theta_0 = \mathbb{E}(\nu_1(\Pi(X)) - \nu_0(\Pi(X))). \tag{5.1}$$

Similar arguments can be made for other measures of program effectiveness (e.g. Heckman, Ichimura, and Todd, 1998). Estimating the ATE by a sample analogue of (5.1) requires nonparametric estimation of the functions $\nu_1(\pi)$ and $\nu_0(\pi)$. Since the propensity score is generally unknown and has to be estimated in a first stage, this fits into our framework with $Z \equiv (Y, X, (D, X))$, $r_0(X_r) \equiv \Pi(X)$, $t(X, r_0(X_r), \theta) \equiv (D, \Pi(X))$, $m_0(z_1) \equiv \nu_d(p)$ and $q(z, \theta, m_0, r_0) \equiv \nu_1(\Pi(x)) - \nu_0(\Pi(x)) - \theta$.

A natural estimate of the ATE is thus the following sample version of (5.1):

$$\hat{\theta} = \frac{1}{n} \sum_{i=1}^{n} (\hat{\nu}_1(\hat{\Pi}(X_i)) - \hat{\nu}_0(\hat{\Pi}(X_i))),$$

where $\hat{\Pi}(x)$ is the q-th order local polynomial estimator of $\Pi(x)$, and $\hat{\nu}_d(\pi)$ is the local linear estimator of $\nu_d(\pi)$, computed using the first-stage estimates of the propensity score (alternatively, we could consider a parametric estimator for the propensity score, such as e.g. Probit). Here the binary covariate D is accommodated via the usual frequency method, i.e. the estimate $\hat{\nu}_d$ is computed by local linear regression of Y_i on $\hat{\Pi}(X_i)$ using the $n_d = \sum_{i=1}^{n} \mathbb{I}\{D_i = d\}$ observations with $D = d$ only. The following proposition asymptotic gives the asymptotic properties of the estimator.[7]

[7]The form of the influence function was also obtained by Hahn and Ridder (2011), who use the approach in Newey (1994) to compute the influence function of the semiparametric estimator $\hat{\theta}$. In contrast to our paper, they do not give conditions for root-n consistency and asymptotic normality of the estimator.

Proposition 1. *Suppose that the regularity conditions given in Appendix B.1 hold. Then we have that* $\sqrt{n}(\hat{\theta} - \theta_0) \xrightarrow{d} N(0, \mathbb{E}(\Psi(Y, D, X)^2)$, *where*

$$\Psi(Y, D, X) = \mu_1(X) - \mu_0(X) + \frac{D(Y - \mu_1(X))}{\Pi(X)} - \frac{(1 - D)(Y - \mu_0(X))}{1 - \Pi(X)} - \theta_0$$

is the influence function, and $\mu_d(x) = \mathbb{E}(Y|D = d, X = x)$ *for* $d = 0, 1$.

Under the conditions of the proposition the asymptotic variance of $\hat{\theta}$ equals the corresponding semiparametric efficiency bound obtained by Hahn (1998). The estimator obtained via regression on the estimated propensity score thus has the same first-order limit properties as other popular efficient estimators of the ATE under unconfoundedness, such as e.g. the propensity score reweighting estimator of Hirano, Imbens, and Ridder (2003).

5.2. Estimation of Production Functions. When estimating the parameters of production functions, a simultaneity problem arises if there is contemporaneous correlation between a firm's inputs and shocks to productivity. In a highly influential paper, Olley and Pakes (1996) propose a methodology to address this issue, which can be seen as a control function approach. Here we consider a simplified version of their method, as described in Levinsohn and Petrin (2003). This setting assumes that firms do not age and cannot be closed. The Cobb-Douglas model for log output Y_t of a firm in period t is given by

$$Y_t = \beta_0 + \beta_L L_t + \beta_K K_t + \omega_t + \eta_t, \tag{5.2}$$

where L_t and K_t are labor and capital inputs, respectively, ω_t is a productivity index that follows a first-order Markov process, and η_t is an i.i.d. productivity shock. Here ω_t and η_t are both unobserved. The main difference is that ω_t is a state variable, and hence impacts the firm's input choices, while η_t has no impact on firm behavior. In particular, the firms' investment I_t in the capital stock is a function of ω_t and K_t: $I_t = \iota_t(\omega_t, K_t)$. Under suitable conditions, firms that choose to invest have investment functions that are strictly increasing in the unobserved productivity index, and hence by invertability ω_t can be written as function of capital and investment

$$\omega_t = \omega(K_t, I_t).$$

Substituting this relationship into (5.2), we find that

$$Y_t = \beta_L L_t + \phi_t + \eta_t, \tag{5.3}$$

where $\phi_t = \phi(K_t, I_t) = \beta_K K_t + \omega(K_t, I_t)$. Equation (5.3) is a standard partially linear model, and thus β_L and the function $\phi(\cdot)$ can be identified and estimated as in Robinson (1988) through the usual least squares arguments. To identify the coefficient β_K, it is assumed that capital does not immediately respond to innovations in the productivity index ω_t, which together with the Markov assumption implies that

$$\omega_t = \Pi(\omega_{t-1}) + \xi_t \text{ with } \mathbb{E}(\xi_t|\omega_{t-1}, K_t) = 0.$$

We can thus rewrite the output net of labor's contribution $Y_t^* = Y_t - \beta_L L_t$ as

$$Y_t^* = \beta_K K_t + \Pi^*(\phi_{t-1} - \beta_K K_{t-1}) + \eta_t^*, \tag{5.4}$$

with $\Pi^*(x) = \Pi(x) + \beta_0$ and $\eta_t^* = \eta_t + \xi_t$. Note that while equation (5.4) resembles a partially linear model (given knowledge of β_L and $\phi(\cdot)$), its structure is actually somewhat different, as the coefficient β_K appears both in the linear part and inside the unknown function Π^*. Still, the parameter β_K can be characterized as the solution to a profiled nonlinear least squares problem:

$$\beta_K = \operatorname*{argmin}_b \mathbb{E}(Y_t - \beta_L L_t - bK_t - \pi(\phi_{t-1} - bK_{t-1}|b))^2, \tag{5.5}$$

where $\pi(c|b) = \mathbb{E}(Y_t - \beta_L L_t - bK_t|\phi_{t-1} - bK_{t-1} = c)$ for any $b \in \mathbb{R}$. Implementing a sample analogue of (5.5) to estimate β_K requires nonparametric estimation of the function $\pi(\cdot|b)$ using an estimates of the coefficient β_L and the function $\phi(\cdot)$, both obtained by estimating (5.3) in a first stage. This problem fits into our framework with $Z \equiv (Y_t, L_t, K_t, I_t, K_{t-1}, I_{t-1})$, $\theta_0 \equiv \beta_K$, $r_0(X_r) \equiv (\beta_L, \phi_{t-1})$, $T(X, \theta, r_0) \equiv \phi_{t-1} - bK_{t-1}$, $m_0(\cdot, \theta) \equiv \pi(\cdot|b)$ and $q(Z, \theta, m_0, r_0) \equiv (Y_t - \beta_L L_t - bK_t - \pi(\phi_{t-1} - bK_{t-1}|b))(K_t - \partial_b \pi(\phi_{t-1} - bK_{t-1}|b)K_{t-1})$.

To give an explicit expression for an estimator $\hat{\beta}_K$ of β_K, let $\hat{\beta}_L$ and $\hat{\phi}(\cdot)$ be estimates of β_L and $\phi(\cdot)$, respectively, obtained via the method in Robinson (1988). For every $b \in \mathbb{R}$, let $\hat{\pi}(\cdot|b)$ be an estimate of $\hat{\pi}(\cdot|b)$, computed by local linear regression of $Y_{it} - \hat{\beta}_L L_{it} - bK_{it}$ on $\hat{\phi}_{i,t-1} - bK_{i,t-1}$. Then we can define the final estimator as

$$\hat{\beta}_K = \operatorname*{argmin}_b \frac{1}{n} \sum_{i=1}^{n} (Y_{it} - \hat{\beta}_L L_{it} - bK_{it} - \hat{\pi}(\hat{\phi}_{i,t-1} - bK_{it-1}|b))^2. \tag{5.6}$$

Note that computing $\hat{\pi}(\cdot|b)$ and $\phi(\cdot)$ involves the use of a generated *dependent* variable. However, compared to the problems arising from the presence of generated covariates, this issue is straightforward to address for linear smoothers like local linear regression. To simplify the expression for the influence function, we introduce the following notation: Let $\pi^{(b)}(c|b) = \partial_a \pi(a|b)|_{a=c} + \partial_a \pi(c|a)|_{a=b}$ be the total derivative of $\pi(b|b)$ with respect to b, and $\pi'(c|b) = \partial_c \pi(c|b)$ the ordinary derivative with respect to the first component. We also define $G_{it} = K_{it} - \pi^{(b)}(\phi_{i,t-1} - \beta_K K_{i,t-1}|\beta_K)K_{i,t-1}$ and the "projection residuals" $G_t^\perp = G_t - \mathbb{E}(G_t|\phi_{t-1} - \beta_K K_{t-1})$ and $L_t^\perp = L_t - \mathbb{E}(L_t|\phi_{t-1} - \beta_K K_{t-1})$.

Proposition 2. *Suppose that the regularity conditions given in Appendix B.2 hold. Then we have that* $\sqrt{n}(\hat{\beta}_K - \beta_K) \xrightarrow{d} N(0,\Omega)$ *with*

$$\Omega = Q_0^{\theta-1}\mathbb{E}\left[(\Psi_0 + \Psi_1 + \Psi_2)(\Psi_0 + \Psi_1 + \Psi_2)^\top\right] Q_0^{\theta-1},$$

where

$$\begin{aligned}
\Psi_0 &= G_t^\perp \eta_t^* \\
\Psi_1 &= -\mathbb{E}(G_t^\perp|K_{t-1}, I_{t-1})\pi'(\phi_{t-1} - \beta_K K_{t-1}|\beta_K)\eta_{t-1} \\
\Psi_2 &= -\mathbb{E}(G_t(L_t^\perp - \mathbb{E}(L_t|K_{t-1}, I_{t-1})\pi'(\phi_{t-1} - \beta_K K_{t-1}|\beta_K))) \\
&\qquad \times \mathbb{E}((L_t - \mathbb{E}(L_t|K_t, I_t))^2)^{-1}(L_t - \mathbb{E}(L_t|K_t, I_t))\eta_t.
\end{aligned}$$

Asymptotic properties of the above estimation procedure were first studied in Pakes and Olley (1995). Our expression for the influence function given in Proposition 2 differs from the their result, even when taking into account that we only consider a simplified version of their model. The reason is that our derivation does account for the estimation error from using an estimate of $\phi(\cdot)$ when *estimating* $\hat{\pi}(\cdot|b)$, and not only for the estimation error resulting from using an estimate of $\phi(\cdot)$ when *evaluating* $\hat{\pi}(\cdot|b)$. In our Proposition 2, both contributions are collected in the term Ψ_1.

A. Proofs of Main Results

A.1. Proof of Theorem 2. To simplify notation, we provide the proof only for the special case $d_T = 1$, i.e. $T = T(X,\theta,r)$ is a univariate random variable, but calculated rates are stated in general form. The proof for higher-dimensional T is conceptionally similar. The following notation is used throughout all our proofs. The unit vector $(1,0,\dots,0)^\top$ in $\mathbb{R}^{p+1}$ is denoted by e_1. We write

$$\begin{aligned} w_i(t,\theta,r) &= (1,(T_i(r,\theta)-t)/h,...,(T_i(r,\theta)-t)^p/h^p)^\top, \\ M_h(t,\theta,r) &= \frac{1}{n}\sum_{i=1}^n w_i(t,r,\theta)w_i(t,r,\theta)^\top K_h(T_i(r,\theta)-t), \\ m_0^*(t,\theta) &= (m_0(t,\theta),hm_0'(t,\theta)/2,...,h^p m_0^p(t,\theta)/p!)^\top, \end{aligned}$$

and $N_h(t,\theta)=\mathbb{E}(M_h(t,\theta))$. Furthermore, we set $w_i(t,\theta)=w_i(t,\theta,r_0)$ and $\hat{w}_i(t,\theta)=w_i(t,\theta,\hat{r})$, and define $M_h(t,\theta)$ and $\widehat{M}_h(t,\theta)$ analogously. Using $\varepsilon^*(\theta)=\varepsilon(\theta)-\rho(X,\theta)$, we can write

$$Y_i = m_0(T_i(\theta),\theta)+\varepsilon_i^*(\theta)+\rho(X_i,\theta)\,.$$

Note that $\mathbb{E}(\varepsilon^*(\theta)|X)=0$ for any $\theta\in\Theta$. With this representation of the dependent variable, we define the following decompositions of both the real and the oracle estimator:

$$\widehat{m}(t,\theta)=m_0(t,\theta)+\widehat{m}_A(t,\theta)+\widehat{m}_B(t,\theta)+\widehat{m}_C(t,\theta)+\widehat{m}_D(t,\theta)+\widehat{m}_E(t,\theta)$$
$$\widetilde{m}(t,\theta)=m_0(t,\theta)+\widetilde{m}_A(t,\theta)+\widetilde{m}_B(t,\theta)+\widetilde{m}_C(t,\theta)+\widetilde{m}_D(t,\theta)+\widetilde{m}_E(t,\theta),$$

with respective components $\widehat{m}_j(t,\theta)=e_1^\top\beta_j(\theta,\widehat{r})$ and $\widetilde{m}_j(t,\theta)=e_1^\top\beta_j(\theta,r_0)$ defined for $j\in\{A,B,C,D,E\}$ as follows:

$$\beta_A(\theta,r)=\operatorname*{argmin}_\beta\sum_{i=1}^n(\varepsilon_i^*(\theta)-\beta^\top w_i(t,\theta,r))^2K_h(T_i(\theta,r)-t),$$
$$\beta_B(\theta,r)=\operatorname*{argmin}_\beta\sum_{i=1}^n(m_0(T_i(\theta,r_0),\theta)-m_0^*(t,\theta)^\top w_i(t,\theta,r_0)-\beta^\top w_i(t,\theta,r))^2K_h(T_i(\theta,r)-t),$$
$$\beta_C(\theta,r)=\operatorname*{argmin}_\beta\sum_{i=1}^n(m_0^*(t,\theta)^\top w_i(t,\theta,r_0)-m_0^*(t,\theta)^\top w_i(t,\theta,r)-\beta^\top w_i(t,\theta,r))^2K_h(T_i(\theta,r)-t),$$
$$\beta_D(\theta,r)=\operatorname*{argmin}_\beta\sum_{i=1}^n(m_0^*(t,\theta)^\top w_i(t,\theta,r)-\beta^\top w_i(t,\theta,r))^2K_h(T_i(\theta,r)-t),$$
$$\beta_E(\theta,r)=\operatorname*{argmin}_\beta\sum_{i=1}^n(\rho(X_i,\theta)-\beta^\top w_i(t,\theta,r))^2K_h(T_i(\theta,r)-t).$$

Finally, we denote the component-wise differences between the real and the oracle estimator by

$$R_{j,n}(t,\theta)=\widehat{m}_j(t,\theta)-\widetilde{m}_j(t,\theta) \text{ for } j\in\{A,B,C,D,E\}. \tag{A.1}$$

The statement of the theorem follows if for any $\theta\in\Theta$ the remainder term $R_n(t,\theta)=\widehat{m}(t,\theta)-\widehat{m}_\Delta(t,\theta)$ satisfies

$$\int R_n(t,\theta)\omega(t)\,\mathrm{d}t=O_P(n^{-\kappa^*})\,.$$

Here $\widehat{m}_\Delta(t,\theta) = \tilde{m}(t,\theta) + \varphi_n^A(t,\theta,\widehat{r}) + \varphi_n^B(t,\theta,\widehat{r})$. The term $\varphi_n^B(t,\theta,r)$ is as defined in (3.2) , and for $p = 1$ the term $\varphi_n^A(t,\theta,r)$ is also as defined in (3.2). More generally, for uneven $p > 1$ we set

$$\varphi_n^A(t,\theta,r) = e_1^\top N_h(\theta)^{-1}\mathbb{E}(K_h(T_i(r)-t)w_i(t,\theta,r)m'_{pol}(T_i(r),t,\theta)(T_i(r,\theta)-T_i(\theta)), \tag{A.2}$$

where $m'_{pol}(u,t,\theta)$ is the derivative of $m_{pol}(u,t,\theta)$ with respect to its first argument and $m_{pol}(u,t,\theta)$ is the following polynomial approximation of $m_0(u,\theta)$ in a neighborhood of t:

$$m_{pol}(u,t,\theta) = m_0^*(t,\theta)^\top(1,(u-t)/h,...,(u-t)^p/(p!h^p))^\top.$$

To simplify the notation, we fix $\theta = \theta_0$ for the rest of the proof and we omit θ as an argument of functions. To prove Theorem 2, we will then show that

$$\int R_{A,n}(t)\omega(t)\,\mathrm{d}t = O_P(n^{-\kappa_1^*}), \tag{A.3}$$

$$\int R_{B,n}(t)\omega(t)\,\mathrm{d}t = O_P(n^{-\kappa_2^*}), \tag{A.4}$$

$$\int R_{C,n}(t)\omega(t)\,\mathrm{d}t = \int \varphi_n^A(t,\hat{r})\omega(t)\,\mathrm{d}t + O_P(n^{-\kappa_3^*}+n^{-\kappa_4^*}), \tag{A.5}$$

$$\int R_{E,n}(t)\omega(t)\,\mathrm{d}t = \int \varphi_n^B(t,\hat{r})\omega(t)\,\mathrm{d}t + O_P(n^{-\kappa_1^*}+n^{-\kappa_2^*}). \tag{A.6}$$

where the terms $R_{n,j}$ are defined in (A.1) above. This directly implies the statement of the theorem since

$$\int (\widehat{m}(t)-\widetilde{m}(t))\omega(t)\,\mathrm{d}t = \sum_{j\in\{A,\dots,E\}} \int R_{n,j}(t)\omega(t)\,\mathrm{d}t, \tag{A.7}$$

and $R_{D,n}(t) \equiv 0$ by construction.

We start with the proof of (A.3). Denote $\Phi_i(t,r) = e_1^\top M_h(t,r)^{-1}w_i(t,r)K_h(T_i(r)-t)$ and write $\Phi_i(r) = \int \Phi_i(t,r)\omega(t)\,\mathrm{d}t$. Furthermore let $L_h(T_i(r)-t) = K_h(T_i(r)-t)w_i(t,r)$ be a vector-valued kernel type function. Then it holds that

$$R_{A,n}(t) = \frac{1}{n}\sum_{i=1}^n (\Phi_i(t,r_0) - \Phi_i(t,\widehat{r}))\,\epsilon_i^*.$$

Using elementary arguments, one can show that

$$M_h(T_i(r_1),r_1) - M_h(T_i(r_2),r_2) = O_P(n^{\eta_{max}})\|r_1-r_2\|_\infty.$$

uniformly for $r_1, r_2 \in \mathcal{R}_n^*$ and $1 \le i \le n$. With the help of this bound, we find that, uniformly for $r_1, r_2 \in \mathcal{R}_n^*$ and $1 \le i \le n$ and some generic constant $c > 0$ which can take different values at each appearance

$$
\begin{aligned}
&|\Phi_i(r_1) - \Phi_i(r_2)| \\
&\leq \left| \int \left[e_1^\top M_h(t, r_1)^{-1} L_h(T_i(r_1) - t) - e_1^\top M_h(t, r_2)^{-1} L_h(T_i(r_2) - t) \right] \omega(t) dt \right| \\
&= \left| \int \left[e_1^\top M_h(T_i(r_1) - hu, r_1)^{-1} \omega(T_i(r_1) - hu) \right. \right. \\
&\qquad \left. \left. - \ e_1^\top M_h(T_i(r_2) - hu, r_2)^{-1} \omega(T_i(r_2) - hu) \right] L(u) du \right| \\
&\leq \max_{1 \leq j \leq d_T} c n^{\eta_j} |T_j(r_1) - T_j(r_2)|. \qquad \text{(A.8)}
\end{aligned}
$$

This last bound can be used to calculate a rough bound on the entropy $H_n(\lambda)$ of the class of functions $i \to \Phi_i(r)$. Using Assumption 3, this class of functions can be covered by $c \exp((\lambda n^{-\eta_j})^{-\alpha} n^{\xi})$ balls of radius $\lambda n^{-\eta_j}$. Thus we find that the entropy $H_n(\lambda) \leq c \max_{1 \leq j \leq d_t} \lambda^{-\alpha_j} n^{\eta_j \alpha_j + \xi_j}$ for some constant $c > 0$. This implies

$$
\int_0^{C_n} H_n^{1/2}(\lambda) d\lambda \leq c n^{-(1-\alpha_{max}/2)\delta_{min} + (\eta\alpha+\xi)_{max}/2}
$$

for $C_n = n^{-\delta_{min}}$. We now apply Theorem 8.13 in van de Geer (2000) with $\bar{Z}_\theta = n^{-1} \sum_{i=1}^n Z_{i,\theta}$, $Z_{i,\theta} = \Phi_i(r)\epsilon_i^*$, $\theta = r$, $R = C_n = n^{-\delta_{min}}$, and a is the entropy bound above. Conditional on observations $X_1, ..., X_n$, we obtain an exponential bound for $\bar{Z}_\theta$ uniformly in $\mathcal{R}_n^*$ since $\frac{1}{n} \sum_{i=1}^n \mathbb{E}[\exp(\ell^* |\epsilon_i^*|) | X_i] \leq C^*$ with probability tending to one, for some constants $C^*, \ell^* > 0$ due to Assumption 1 (iv). With standard arguments this yields

$$
\sup_{r_1, r_2 \in \mathcal{R}_n^*} \frac{1}{n} \sum_{i=1}^n (\Phi_i(r_1) - \Phi_i(r_2)) \epsilon_i^* = o_P \left(n^{-(1/2) - (1-\alpha_{max}/2)\delta_{min} + (\eta\alpha+\xi)_{max}/2} \right). \qquad \text{(A.9)}
$$

Equation (A.9) provides the desired result (A.3) for R_A.

For the proof of (A.4), note that for some nonnegative integers a, b and constants $C_1, C_2 > 0$ it holds that $\left| m_0(T_i(r)) - m_0^*(t)^\top w_i(t, r) \right| \leq C_1 n^{-(p+1)\eta_{min}}$ and

$$
\left| \frac{1}{n} \sum_{i=1}^n K_h(T_i(r_1) - t) w_{i,k}^a(t, r_1) w_{i,l}^b(t, r_1) - K_h(T_i(r_2) - t) w_{i,k}^a(t, r_2) w_{i,l}^b(t, r_2) \right| \leq C_2 n^{-(\delta-\eta)_{min}}
$$

for components l, k and all $t \in I_T^*$ and $r, r_1, r_2 \in \mathcal{R}_n^*$. These two statements directly imply (A.4).

For the proof of (A.5), note that uniformly over $1 \leq i \leq n$ and $r \in \mathcal{R}_n^*$ it holds that

$$
m_0^*(t)^\top w_i(t, r_0) - m_0^*(t)^\top w_i(t, r) = m'_{pol}(T_i(\theta), t)(T_i(r) - T_i(r_0)) + O_P(n^{-2\delta_{min}}).
$$

Substituting this expression into $R_{C,n}$, we find that

$$
\int R_{C,n}(t) \omega(t) dt = \frac{1}{n} \sum_{i=1}^n \Phi_i^*(\widehat{r})(T_i(\widehat{r}) - T_i(r_0)) + O_P(n^{-2\delta_{min}}),
$$

where

$$\Phi_i^*(r) = \int e_1^\top M_h(t,r)^{-1} L_h(T_i(r)-t) m'_{pol}(T_i(r),t)\omega(t)dt.$$

Furthermore, we have that

$$\int \varphi_n^A(t,\widehat{r})\omega(t)dt = \frac{1}{n}\sum_{i=1}^n \Phi_i^*(r_0)(T_i(\widehat{r}) - T_i(r_0)) + o_p(n^{-1/2}).$$

Thus, for (A.5) we have to show that

$$\frac{1}{n}\sum_{i=1}^n (\Phi_i^*(\widehat{r}) - \Phi_i^*(r_0))(T_i(\widehat{r}) - T_i(r_0)) = O_P(n^{-\kappa_3^*} + n^{-\kappa_4^*}). \tag{A.10}$$

Since $|T_i(\widehat{r}) - T_i(r_0)| = O_P(n^{-\delta_{min}})$ uniformly over $r \in \mathcal{R}_n^*$ and $1 \le i \le n$, one only has to prove that

$$|\Phi_i^*(r) - \Phi_i^*(r_0)| = O_P(n^{-\kappa_4^* + \delta_{min}} + n^{-\delta_{min}})$$

that uniformly for $r \in \mathcal{R}_n^*$ and $1 \le i \le n$ in order to establish (A.10). To see why the last claim holds, note that we can write:

$$\begin{aligned}
\Phi_i^*(r) - \Phi_i^*(r_0) &= \int e_1^\top [M_h(t,r)^{-1} L_h(T_i(r)-t) m'_{pol}(T_i(r),t) \\
&\quad - M_h(t,r_0)^{-1} L_h(T_i(r_0)-t) m'_{pol}(T_i(r_0),t)]\omega(t)dt \\
&= \int e_1^\top [M_h(T_i(r)-hu,r)^{-1}\omega(T_i(r)-hu) m'_{pol}(T_i(r),T_i(r)-hu) \\
&\quad - M_h(T_i(r_0)-hu,r_0)^{-1}\omega(T_i(r_0)-hu) m'_{pol}(T_i(r_0),T_i(r_0)-hu)] L(u)du.
\end{aligned}$$

First, it is easy to see that

$$\max_{1\le i\le n}\sup_{r\in\mathcal{R}_n^*}\sup_{t\in I_T^*} |\omega(T_i(r)-t) - \omega(T_i(r_0)-t)| = O_P(n^{-\delta_{min}}) \quad \text{and}$$

$$\max_{1\le i\le n}\sup_{r\in\mathcal{R}_n^*}\sup_{t\in I_T^*} |m'_{pol}(T_i(r),T_i(r)-t) - m'_{pol}(T_i(r_0),T_i(r_0)-t)| = O_P(n^{-\delta_{min}})$$

due to the smoothness of the functions involved. It thus remains to consider the elements of the matrix $M_h(T_i(r)-t,r) - M_h(T_i(r_0)-t,r_0)$. Any such element is of the form

$$\frac{1}{n}\sum_{i=1}^n \big[(T_i(r)-t)^u h^{-u} K_h(T_i(r)-t)\big] - \big[(T_i(r_0)-t)^u h^{-u} K_h(T_i(r_0)-t)\big]$$

for some $0 \le u_+ \le p$. We thus show that

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^n \big[(T_i(r)-t)^u h^{-u} K_h(T_i(r)-t)\big] \\
&\qquad - \big[(T_i(r_0)-t)^u h^{-u} K_h(T_i(r_0)-t)\big] = O_P(n^{-\delta_{min}} + n^{-\kappa_4^* + \delta_{min}}).
\end{aligned} \tag{A.11}$$

uniformly over $r \in \mathcal{R}_n^*$. Because of Assumption 4(iii), we have that

$$\mathbb{E}\left[(T_i(r)-t)^u h^{-u} K_h(T_i(r)-t)\right] - \mathbb{E}\left[(T_i(r_0)-t)^u h^{-u} K_h(T_i(r_0)-t)\right] = O_P(n^{-\delta_{min}}).$$

uniformly over $r \in \mathcal{R}_n^*$. Thus, for a proof of (A.11) it suffices to establish that

$$\frac{1}{n}\sum_{i=1}^{n}\left[(T_i(r)-t)^u h^{-u} K_h(T_i(r)-t)\right] - \mathbb{E}\left[(T_i(r)-t)^u h^{-u} K_h(T_i(r)-t)\right]$$
$$- \left[(T_i(r_0)-t)^u h^{-u} K_h(T_i(r_0)-t)\right] - \mathbb{E}\left[(T_i(r_0)-t)^u h^{-u} K_h(T_i(r_0)-t)\right] = O_P(n^{-\kappa_4^*+\delta_{min}}).$$

The last claim follows from the same type of arguments used in the treatment of $R_{A,n}$. Taken together, the above derivation shows that

$$\int R_{C,n}(t)\omega(t)\,\mathrm{d}t = \int \varphi_n^A(t,\hat{r})\omega(t)\,\mathrm{d}t + O_P(n^{-\kappa_4^*} + n^{-\kappa_5^*}),$$

as claimed

It remains to show (A.6). Note that

$$\int R_{E,n}(t)\omega(t)\,\mathrm{d}t = \frac{1}{n}\sum_{i=1}^{n}[\Phi_i(\hat{r}) - \Phi_i(r_0)]\rho(X_i).$$

Using the same reasoning as in the treatment of $R_{A,n}$ and Assumption 4(i)–(ii), we find that

$$\frac{1}{n}\sum_{i=1}^{n}\Phi_i(r)(\rho(X_i) - \mathbb{E}[\rho(X_i)|T_i(r)]) - \Phi_i(r_0)(\rho(X_i) - \mathbb{E}[\rho(X_i)|T_i(r_0)]) = O_P(n^{-\kappa_1^*})$$

uniformly for $r \in \mathcal{R}_n^*$. Note that $\mathbb{E}[\rho(X_i)|T_i(r_0)] = 0$. We now use that

$$\frac{1}{n}\sum_{i=1}^{n}\Phi_i(r)\mathbb{E}[\rho(X_i)|T_i(r)] = \frac{1}{n}\sum_{i=1}^{n}\int e_1^\top M_h(t,r)^{-1}L_h(T_i(r)-t)\mathbb{E}[\rho(X_i)|T_i(r)]\omega(t)dt$$
$$= \int \varphi_n^B(t)\omega(t)dt + O_P(n^{-\kappa_2^*})$$

uniformly over $r \in \mathcal{R}_n^*$, and thus (A.6) holds. This concludes the proof of Theorem 2. □

A.2. Proof of Theorem 3. First, standard results in e.g. Masry (1996), imply that the oracle estimator $\widetilde{m}$ satisfies

$$\sup_{t\in I_T^*,\theta\in\Theta} |\widetilde{m}(t,\theta) - m_0(t,\theta)| = o_p\left(n^{-p\eta_{min}} + \sqrt{\log(n)n^{-(1-\eta_+)}}\right).$$

uniformly over $t \in I_T^*$ and $\theta \in \Theta$ under the conditions of the theorem. Second, one can show that

$$\sup_{t\in I_T^*,\theta\in\Theta} |\widehat{m}(t,\theta) - \widehat{m}_\Delta(t,\theta)| = o_p(n^{-\kappa}). \tag{A.12}$$

The statement (A.12) is an extension of Theorem 1 in Mammen, Rothe, and Schienle (2011), which gives a stochastic expansion of a local linear estimator regression estimator with generated covariates, and the special case that $T(x, r, \theta) = r(x_r)$. Generalizing this result to higher order local polynomials and more general forms of T is conceptionally straightforward, and thus a proof is omitted. With (A.12), the statement of the Theorem follows from a trivial bound on the leading terms of the expansion $\widehat{m}_\Delta$. □

Remark 5. One could use the additional structure implied by Assumption 5 to prove a somewhat better uniform rate of consistency under some minor additional regularity conditions. In particular, one can show that

$$\sup_{t \in I_T^*, \theta \in \Theta} |\widehat{m}_\Delta(t, \theta) - \widetilde{m}(t, \theta)| = O_P(n^{-\delta_{min}} \sqrt{n^{-(1-\eta_+)} \log n} + n^{-2\delta_{min}}), \tag{A.13}$$

which is better than the rate of $O_P(n^{-\delta_{min}})$ obtained from a crude bound that appears in Theorem 3.

A.3. Proof of Corollary 1. To prove this result, we first establish a linear stochastic expansion for the oracle estimator $\widetilde{m}$. Using arguments in Masry (1996), Kong, Linton, and Xia (2010) or Ichimura and Lee (2010), one can show that

$$\widetilde{m}(t, \theta) = \frac{1}{n} \sum_{i=1}^{n} \varphi^{\tilde{m}}(t, \theta) + O(n^{-p\eta_{min}}) + O_p(\log(n) n^{-(1-\eta_+)}),$$

uniformly over $t \in I_T^*$ and $\theta \in \Theta$, where

$$\varphi_{ni}^{\tilde{m}}(t, \theta) = e_1^\top N_h(t)^{-1} w(T_i(\theta) - t) K_h(T_i(\theta) - t) \varepsilon_i(\theta).$$

with $w(t) = (1, t, ..., t^p)^\top$ and $N_h(t, \theta) = \mathbb{E}(w((T_i(\theta) - t)/h, \theta) w((T(\theta) - t)/h, \theta)^\top K_h(T(\theta) - t))$. Next, note that the conditions of the corollary imply that that $O(n^{-p\eta_{min}}) = o(n^{-1/2})$ and $O_p(\log(n) n^{-(1-\eta_+)}) = o_p(n^{-1/2})$ and $O(n^{-2\delta_{min}}) = o_p(n^{-1/2})$. Applying Theorem 2, we therefore we find that Q_0^ξ can be decomposed as follows:

$$Q_0^\xi[\hat{\xi} - \xi_0] = A_1 + A_2 + A_3 + A_4 + o_p(n^{-1/2}),$$

where

$$\begin{aligned}
A_1 &= \int \lambda_m(z_m)\frac{1}{n}\sum_{i=1}^{n}\varphi_{ni}^{\tilde{m}}(z_m,\theta_0)f_{Z_m}(z_m)dz_m,\\
A_2 &= \int \lambda_m(z_m)\varphi_n^{A}(z_m,\theta_0,\hat{r})f_{Z_m}(z_m)dz_m,\\
A_3 &= \int \lambda_m(z_m)\varphi_n^{B}(z_m,\theta_0,\hat{r})f_{Z_m}(z_m)dz_m\\
A_4 &= \int \lambda_r(x_r)\varphi_{ni}^{\hat{r}}(x_r,\theta_0,\hat{r})f_{X_r}(x_r)dx_r,
\end{aligned}$$

We deal with each of these four terms separately. First, applying standard arguments from kernel smoothing theory, we find that

$$\begin{aligned}
A_1 &= \frac{1}{n}\sum_{i=1}^{n}\varepsilon_i\int e_1^\top N_h(z_m)^{-1}w(T_i(\theta)-z_m)K_h(T_i(\theta)-z_m)\lambda_m(z_m)f_{Z_m}(z_m)dz_m\\
&= \frac{1}{n}\sum_{i=1}^{n}\varepsilon_i\int e_1^\top N_h(T_i-th)^{-1}w(t)K(t)\lambda_m(T_i-th)f_{Z_m}(T_i-th)dt\\
&= \frac{1}{n}\sum_{i=1}^{n}\varepsilon_i\lambda_m(T_i)\frac{f_{Z_m}(T_i)}{f_T(T_i)}+O(n^{-p\eta_{min}})\\
&= \frac{1}{n}\sum_{i=1}^{n}\psi_1(Z_i)+o_p(n^{-1/2})
\end{aligned}$$

For the second term, first note that it follows from standard bias calculations for kernel-type estimators that

$$\begin{aligned}
&\int \lambda_m(z_m)\varphi_n^{A}(z_m,\theta_0,r)f_{Z_m}(z_m)dz_m\\
&= -\mathbb{E}\left(T_i^{(r)}(X)(r(X_{ri})-r_0(X_{ri}))\lambda_m(T_i)m_0'(T_i)\frac{f_{Z_m}(T_i)}{f_T(T_i)}\right)+O_p(h^p)
\end{aligned}$$

uniformly for fixed functions $r \in \mathcal{R}_n^*$. Substituting the expansion for $\widehat{r}-r_0$ from Assumption 5 we then directly find that

$$\begin{aligned}
A_2 &= -\frac{1}{n}\sum_{i=1}^{n}\nu(W_i)\mathbb{E}\left(T^{(r)}(X)\lambda_m(T)m_0'(T)\frac{f_{Z_m}(T)}{f_T(T)}\mathcal{H}_n(S_i,X_r)\middle| S_i\right)\\
&\quad + O_p(n^{-p\eta_{min}}+n^{-2\delta_{min}})+o_p(n^{-1/2})\\
&= \frac{1}{n}\sum_{i=1}^{n}\psi_2^A(Z_i)+o_p(n^{-1/2}).
\end{aligned}$$

Concerning the term A_3, we have that

$$
\begin{aligned}
A_3 &= \iint \frac{\lambda_m(z_m)}{f_T(z_m)} K_h'(T(x)-z_m)(\hat{T}(x)-T(x))\rho(x) f_{Z_m}(z_m) f_X(x)\,\mathrm{d}x dz_m \\
&= \int \frac{1}{h}\int K'(t)G(T(x)+th)\,\mathrm{d}t(\hat{T}(x)-T(x))\rho(x)f_X(x)\,\mathrm{d}x \\
&= \int G'(T(x))(\hat{T}(x)-T(x))\rho(x)f_X(x)\,\mathrm{d}x + O(h^p) \\
&= \int G'(T(x))T^{(r)}(x)\left(\frac{1}{n}\sum_{i=1}^n \mathcal{H}_n(S_i,x_r)\nu(W_i)\right)\rho(x)f_X(x)\,\mathrm{d}x + O_P(h^p + n^{-2\delta_{min}}) \\
&= \frac{1}{n}\sum_{i=1}^n \nu(W_i)\mathbb{E}(G'(T)T^{(r)}(X)\mathcal{H}_n(S_i,X_r)\rho(X)|S_i) + O_P(n^{p\eta_{min}} + n^{-2\delta_{min}}) \\
&= \frac{1}{n}\sum_{i=1}^n \psi_2^B(Z_i) + o_p(n^{-1/2})
\end{aligned}
$$

with $G(t) = \lambda_m(t)f_{Z_m}(t)f_T(t)^{-1}$ and $G'(t) = \partial_t G(t)$ using integration by parts to obtain the fourth equality. Finally, we have

$$
\begin{aligned}
A_4 &= \nu(W_i)\mathbb{E}(\lambda_r(X_r)\mathcal{H}_n(S_i,X_r)|S_i) + o_p(n^{-1/2}) \\
&= \frac{1}{n}\sum_{i=1}^n \psi_3(Z_i) + o_p(n^{-1/2})
\end{aligned}
$$

using the same type of arguments as the ones applied above. The statement of the corollary then follows since $\psi_2 = \psi_2^A + \psi_2^B$.

A.4. Derivation of Example 1. Suppose that r_0 is a q-times continuously differentiable regression function estimated by qth order local polynomial regression using a bandwidth g and a kernel function L. Assume that S is continuously distributed with compact support I_S, and that the corresponding density f_S is q-times continuously differentiable, bounded, and bounded away from zero on I_S. Then it follows under some further standard regularity conditions (e.g. Kong, Linton, and Xia, 2010) that

$$
\hat{r}(s) - r_0(s) = \frac{1}{n}\sum_{i=1}^n e_1^\top N_h^S(s)^{-1} w(S_i - s)L_g(S_i - s)\zeta_i + O_p(g^q + \log(n)/(ng^{d_s}))
$$

uniformly over $s \in I_S$, $w(t) = (1, t, ..., t^p)^\top$ as above and $N_h^S(t) = \mathbb{E}(w((S_i - s)/g,\theta)w((S_i - s)/g,\theta)^\top L_g(S_i - s)$. The remainder term in the last equation can be made as small as $o_p(n^{-1/2})$ by choosing an appropriate bandwidth if q is sufficiently large. It follows that Assumption 5 is satisfied with $\nu(W_i) = \zeta_i$ and $\mathcal{H}_n(S_i, s) = e_1^\top N_h^S(s)^{-1}w(S_i - s)L_g(S_i - s)$. The condition that $\mathbb{E}(\|\mathcal{H}_n(S_i,S_j)\|^2) = o(n)$ holds if $ng^{d_s} \to \infty$. To obtain the explicit expressions for ψ_2 and ψ_3, we insert the above relation into the expression from Corollary 1 and apply standard U-Statistics arguments (e.g. Powell, Stock, and Stoker, 1989). □

A.5. Derivation of Example 2. This derivation is trivial and thus omitted. □

B. Details on Econometric Applications

B.1. Regression on the Propensity Score. In this section, we give details on the construction of the estimator $\hat{\theta}$, and the regularity conditions under which Proposition 1 is valid. The data consist of a sample $\{(Y_i, D_i, X_i), i = 1, \dots, n\}$ from the distribution of (Y, D, X). The estimator of the propensity score $\Pi(x) = \mathbb{E}(D|X = x)$ is given by $\widehat{\Pi}(x) = \widehat{\alpha}$, where

$$(\hat{\alpha}, \hat{\beta}) = \underset{\alpha,\beta}{\operatorname{argmin}} \sum_{i=1}^{n} (D_i - \alpha - \sum_{1 \le u_+ \le q} \beta_u^\top (X_i - x)^u)^2 L_g(X_i - x)$$

and $L_g(s) = \prod_{j=1}^{p} \mathcal{L}(s_j/g)/g$ is a d_x-dimensional product kernel built from the univariate kernel $\mathcal{L}$, g is a bandwidth, which for simplicity is assumed to be the same for all components, and $\sum_{1 \le u_+ \le q}$ denotes the summation over all $u = (u_1, \dots, u_p)$ with $1 \le u_+ \le q$. Next, for $d \in \{0, 1\}$ the estimate of $\nu_d(\pi) = \mathbb{E}(Y|D = d, \Pi(X) = \pi)$ is given by the third-order local polynomial estimator: we set $\widehat{\nu}_d(\pi) = \widehat{\alpha}_d$, where

$$(\widehat{\alpha}_d, \widehat{\beta}_d) = \underset{\alpha,\beta}{\operatorname{argmin}} \sum_{i=1}^{n} \mathbb{I}\{D_i = d\}(Y_i - \alpha - \sum_{1 \le v \le 3} \beta_v^\top (\widehat{\Pi}(X_i) - \pi)^v)^2 K_h(\widehat{\Pi}(X_i) - \pi) ,$$

with $K_h(u) = K(u/h)/h$, K a one-dimensional kernel function and h a bandwidth that tends to zero as the sample size n tends to infinity. The final estimator of θ_0 is then given by

$$\hat{\theta} = \frac{1}{n} \sum_{i=1}^{n} (\hat{\nu}_1(\hat{\Pi}(X_i)) - \hat{\nu}_0(\hat{\Pi}(X_i))).$$

To prove Proposition 1, we make the following assumptions.

Assumption 6. *The sample observations $\{(Y_i, D_i, X_i), i = 1, \dots, n\}$ are i.i.d.*

Assumption 7. *(i) The random vector X is continuously distributed with compact support I_X. Its density function f_X is bounded and bounded away from zero on I_X, and is also $q + 1$-times continuously differentiable for some uneven number $q \ge d_X$. (ii) The function $\Pi(x)$ is bounded away from zero and one on I_X, and is also $q + 1$-times continuously differentiable. (iii) For any $d \in \{0, 1\}$, the random variable $\Pi(X)$ is continuously distributed conditional on $D = d$, with compact support I_Π. Its conditional density function $f_{\Pi|D}(\cdot, d)$ is bounded and bounded. away from zero on I_Π, and is also four times continuously differentiable. (iv) For any $d \in \{0, 1\}$, the function $\nu_d(\pi)$ is four times continuously differentiable on I_Π.*

Assumption 8. *The residual* $\varepsilon = Y - \mathbb{E}(Y|\Pi(X))$ *satisfies* $E[\exp(l|\varepsilon|)|X] \leq C$ *almost surely for a constant* $C > 0$ *and* $l > 0$ *small enough.*

Assumption 9. *(i) The function* K *is twice continuously differentiable and satisfies the following conditions:* $\int K(u)du = 1$, $\int uK(u)du = 0$, $\int |u^2K(u)|du < \infty$, *and* $K(u) = 0$ *for values of* u *not contained in some compact interval, say* $[-1, 1]$. *(ii) The function* $\mathcal{L}$ *is* k*-times continuously differentiable for some natural number* $k \geq \max\{2, d_x/2\}$, *and satisfies the following conditions:* $\int \mathcal{L}(u)du = 1$, $\int u\mathcal{L}(u)du = 1$, *and* $\mathcal{L}(u) = 0$ *for values of* u *not contained in some compact interval, say* $[-1, 1]$.

Assumption 10. *The bandwidths satisfy* $h \sim n^{-\eta}$ *and* $g \sim n^{-\gamma}$ *with* $\gamma = 1/(2q + 1)$ *and* $1/8 < \eta < (q + 2)/(8q + 4)$.

Proof of Proposition 1. The proof uses the same arguments as that of Corollary 1 and Example 1, and thus the details are omitted. The only issue is to show that $\kappa^* > 1/2$. To see this, note that the conditions of the Proposition imply that Assumption 2 holds with $\delta = (q + 1)/(4q + 2) > 1/4$, and that Assumption 3 holds with $\alpha \leq q/(q + 1)$ and $\xi = 0$. The restrictions on η then ensure that $\delta - \eta > (1/2)(\delta\alpha + \xi)$ and $(1 - \eta)/2 - \eta > (1/2)(\delta\alpha + \xi)$. We then easily see that $\kappa^* > 1/2$. □

B.2. Estimation of Production Functions. In this section, we give details on the construction of the estimator $\hat{\theta}$, and the regularity conditions under which Proposition 2 is valid. The data consist of a sample $\{(Y_{it}, L_{it}, K_{it}, I_{it}, K_{it-1}, I_{it-1}), i = 1, \ldots, n\}$ from the distribution of $(Y_t, L_t, K_t, I_t, K_{t-1}, I_{t-1})$. As a first step, we obtain an estimator $\hat{\beta}_L$ of β_L using the method in Robinson (1988). Under regularity conditions given in that paper,

$$\sqrt{n}(\hat{\beta}_L - \beta_L) = \mathbb{E}((L_t - \mathbb{E}(L_t|K_t, I_t))^2)^{-1}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(L_{it} - \mathbb{E}(L_{it}|K_{it}, I_{it}))\eta_{it} + o_p(1).$$

Next, the estimator of $\phi(\cdot)$ is given by $\hat{\phi}(a, b) = \hat{\alpha}$, where

$$(\hat{\alpha}, \hat{\beta}) = \underset{\alpha,\beta}{\operatorname{argmin}} \sum_{i=1}^{n}((Y_{it} - \hat{\beta}_L L_{it}) - \alpha - \sum_{1\leq u_+\leq q} \beta_u^T((K_{it}, I_{it}) - (a, b))^u)^2 L_g((K_{it}, I_{it}) - (a, b)),$$

and $L_g(s) = \prod_{j=1}^{p} \mathcal{L}(s_j/g)/g$ is a d_x-dimensional product kernel built from the univariate kernel $\mathcal{L}$, g is a bandwidth, which for simplicity is assumed to be the same for all components, and $\sum_{1\leq u_+\leq q}$ denotes the summation over all $u = (u_1, \ldots, u_p)$ with $1 \leq u_+ \leq q$. To simplify the exposition below, we also define an infeasible estimator of $\phi(\cdot)$ that uses the true value of the dependent variable. We set $\hat{\phi}^*(a,b) = \hat{\alpha}$, where

$$(\hat{\alpha}, \hat{\beta}) = \underset{\alpha,\beta}{\operatorname{argmin}} \sum_{i=1}^{n} ((Y_{it} - \beta_L L_{it}) - \alpha - \sum_{1 \le u_+ \le q} \beta_r^T ((K_{it}, I_{it}) - (a,b))^u)^2 L_g((K_{it}, I_{it}) - (a,b)).$$

We also define $\hat{\phi}_t = \hat{\phi}(K_t, L_t)$. Next, for every b the estimator of $\pi(\cdot|b)$ is given by the third-order local polynomial estimator $\hat{\pi}(c|b) = \hat{\alpha}$, where

$$(\widehat{\alpha}, \widehat{\beta}) = \underset{\alpha,\beta}{\operatorname{argmin}} \sum_{i=1}^{n} ((Y_{it} - \hat{\beta}_L L_{it} - bK_{it}) - \alpha - \sum_{1 \le v \le 3} \beta_v^\top (\hat{\phi}_{it-1} - bK_{it-1} - c)^v)^2 K_h(\hat{\phi}_{it-1} - bK_{it-1} - c) \,,$$

with $K_h(u) = K(u/h)/h$, K a one-dimensional kernel function, and h a bandwidth that tends to zero as the sample size n tends to infinity. Again, we also define an infeasible estimator that uses the true value of the dependent variable. We set $\hat{\pi}^*(c|b) = \hat{\alpha}$, where

$$(\widehat{\alpha}, \widehat{\beta}) = \underset{\alpha,\beta}{\operatorname{argmin}} \sum_{i=1}^{n} ((Y_{it} - \beta_L L_{it} - bK_{it}) - \alpha - \sum_{1 \le v \le 3} \beta_v^\top (\hat{\phi}_{it-1} - bK_{it-1} - c)^v)^2 K_h(\hat{\phi}_{it-1} - bK_{it-1} - c) \,,$$

Our final estimator is then given as a solution to an empirical moment condition. Let

$$M_n(b) = \frac{1}{n} \sum_{i=1}^{n} (Y_{it} - \hat{\beta}_L L_{it} - bK_{it} - \hat{\pi}(\phi_{it-1} - bK_{it-1}|b))(K_{it} - \partial_b \hat{\pi}(\phi_{it-1} - bK_{it-1}|b)K_{it-1})$$

Then the final estimator $\hat{\beta}_K$ satisfies $M_n(\hat{\beta}_K) = 0$.

To prove Proposition 2, we make the following assumptions.

Assumption 11. *The sample observations* $\{(Y_{it}, L_{it}, K_{it}, I_{it}, K_{it-1}, I_{it-1}), i = 1, \ldots, n\}$ *are i.i.d.*

Assumption 12. *The regularity conditions imposed in Robinson (1988), which ensure that*

$$\sqrt{n}(\hat{\beta}_L - \beta_L) = \mathbb{E}((L_t - \mathbb{E}(L_t|K_t, I_t))^2)^{-1} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (L_{it} - \mathbb{E}(L_{it}|K_{it}, I_{it}))\eta_{it} + o_p(1) \qquad \text{(B.1)}$$

hold.

Assumption 13. *(i) The random vector* $S_{t-1} = (K_{t-1}, I_{t-1})$ *is continuously distributed with compact support* I_S*. Its density function* f_S *is bounded and bounded away from zero on* I_S*, and is also* $q+1$*-times continuously differentiable for some uneven number* $q \ge 3$*. (ii) The function* $\phi(s)$ *is* $q+1$*-times continuously differentiable. (iii) Suppose that* $\beta_K \in \int(B)$ *for some known compact set* B*. For any* $b \in B$*, the random variable* $T_{t-1}(b) = \phi(S_{t-1}) - bK_{t-1}$ *is continuously distributed with compact support* I_T*. Its density function* $f_T(\cdot, b)$ *is bounded and bounded away from zero on* I_T*, uniformly over* $b \in B$*. The density is also four times continuously differentiable. (iv) For any* $b \in B$*, the function* $\pi(\cdot, b)$ *is four times continuously differentiable on* I_T*.*

Assumption 14. *For any $b \in B$, the residual $\varepsilon(b) = (Y_t - \beta_L L_t - bK_t) - \pi(T_{t-1}(b)|b)$ satisfies $E[\exp(l|\varepsilon(b)|)|S_{t-1}] \leq C$ almost surely for a constant $C > 0$ and $l > 0$ small enough.*

Assumption 15. *(i) The function K is two times continuously differentiable and satisfies the following conditions: $\int K(u)du = 1$, $\int uK(u)du = 0$, $\int |u^2 K(u)|du < \infty$, and $K(u) = 0$ for values of u not contained in some compact interval, say $[-1, 1]$. (ii) The function $\mathcal{L}$ is k-times continuously differentiable for some natural number $k \geq 2$, and satisfies the following conditions: $\int \mathcal{L}(u)du = 1$, $\int u\mathcal{L}(u)du = 1$, and $\mathcal{L}(u) = 0$ for values of u not contained in some compact interval, say $[-1, 1]$.*

Assumption 16. *The bandwidths satisfy $h \sim n^{-\eta}$ and $g \sim n^{-\gamma}$ with $\gamma = 1/(2q + 1)$ and $1/8 < \eta < (q + 2)/(8q + 4)$.*

Proof of Proposition 2. Again, we can use the same arguments as that of Corollary 1 and Example 1 to show this result. To show that $\kappa^* > 1/2$ under the conditions of the proposition, we proceed as in the proof of Proposition 1. To derive the influence function, it is useful to note that (4.2)–(4.3) hold with

$$\lambda_m(c) = -\mathbb{E}(G_t|T_{t-1} = c)$$
$$\lambda_r(c_1, c_2) = -(\mathbb{E}(\pi'(T_{t-1})G_t|S_{t-1} = c_1), \mathbb{E}(G_t|L_t = c_2))^\top.$$

Moreover, the proof uses that

$$\hat{\phi}_{t-1} = \hat{\phi}^*_{t-1} - \mathbb{E}(L_{t-1}|K_{t-1}, I_{t-1})(\hat{\beta}_L - \beta_0) + o_p(n^{-1/2})$$
$$\hat{\pi}(c|b) = \hat{\pi}^*(c|b) - (\hat{\beta}_L - \beta_L)\mathbb{E}(L_t|\phi_{t-1} - bK_{t-1} = c) + o_p(n^{-1/2}).$$

This follows directly from the linearity of the local polynomial smoothing operator. □

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