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ABSTRACT

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Players cooperate in experiments more than game theory would predict. In order to explain this, we introduce the 'returns-based beliefs' approach: the expected returns of a particular strategy in proportion to the total expected returns of all strategies. Using a decision analytic solution concept, Luce's (1959) probabilistic choice model, and 'hyperpriors' for ambiguity in players' cooperability, our approach explains empirical observations in classic games such as the Prisoner's Dilemma. Testing the closeness of fit of our model on Selten and Chmura (2008) data for completely mixed 2x2 games shows that with loss aversion, returns-based beliefs explain the data better than other equilibrium concepts.

JEL Classification: D01, D03

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I. INTRODUCTION

Scholars in statistics and economics have highlighted a number of game-theoretic contradictions and paradoxes in which individual decision-making in real-world situations is at odds with what is predicted by game theory (Goeree and Holt 2001; Luce and Raiffa 1957; Selten 1978; Rios, Rios and Banks 2009; Rosenthal 1981). For example, one of the most widely analyzed games in economics, mathematics, biology and other sciences is the Prisoner's Dilemma, a two-by-two noncooperative game. The Prisoner's Dilemma lies at the heart of important concepts in game theory such as the 'Nash equilibrium' (Nash 1951). Empirical tests demonstrate that in the real world people are often more cooperative than that predicted by the outcome of the Prisoner's Dilemma. Many non-Nash equilibrium concepts have been proposed to explain these anomalies. In this paper we provide a reason for this unreason[1]: we propose an alternative equilibrium concept to the Nash by which people might form their beliefs to play their strategies which we call '*returns-based beliefs*'. We show that this might explain better cooperation in the Prisoner's Dilemma. We also test the closeness of fit of our returns-based beliefs model with hyperpriors, and by incorporating loss aversion, using data from Selten and Chmura (2008) for completely mixed 2×2 games. In doing so, we show that the returns based belief model is able to explain the data better than other stationary concepts.

In Nash equilibrium, each player chooses the action that maximizes their returns subject to the opponent's choice and no player can gain by changing their strategy unilaterally. Therefore, in the Nash equilibrium the players form beliefs in such a way as to be consistent with the actions being in equilibrium. In discussing risk analysis and game theory in a recent paper on *Adversarial Risk Analysis*, Rios, Rios and Banks (2009, pp. 845-849) provide a discussion of the difference between game theoretic and decision analytic solution concepts. In this paper, we also use a decision analytic approach where individuals form subjective beliefs over the actions of their opponent and choose a mixed strategy profile

[1] We attribute this expression to Herbert Simon (1987) who wrote, 'Sometimes the term rational (or logical) is applied to decision making that is consciously analytic, the term non-rational to decision making that is intuitive and judgmental, and the term irrational to decision making and behaviour that responds to the emotions or that deviates from action chosen "rationally". We will be concerned then, with the non-rational and the irrational components of managerial decision making and behaviour. Our task, you might say, is to discover the reason that underlies unreason.' (Simon, 1987, p.57)

over the actions based on the relative returns[2]. In our approach, we use the probabilistic choice model developed axiomatically by Luce (1959) as the basis of forming the subjective beliefs. Returns-based beliefs bring squarely into the picture the emphasis on the relative attractiveness of individual actions in choosing the optimal strategy. Our approach treats *beliefs* rather than *strategies* as the primary concept (p.139, Binmore 2009; Nau and McCardle 1990). In doing so, we assume that players' subjective beliefs are in equilibrium with reference to each other. Therefore, consistent with a decision theoretic perspective, players adopt strategies on the basis of their respective subjective beliefs.

In the returns-based beliefs model, we propose three modifications to standard Nash equilibrium for finite normal form games. The first is to replace best replying by each agent with Luce's (1959) probabilistic choice model. We assume that the primitive of choice is a probability that captures the likelihood of one action being chosen over the other based on the expected utilities[3] (Blavatskyy 2008). As a result the players have a probabilistic best response. This can be understood as assuming that players choose probabilistically (with higher payoff actions receiving higher probability), while they hold correct beliefs about the choices of their opponents. The second modification to the Nash equilibrium is to introduce probabilistic beliefs. We assume that the players' beliefs about the choice of other players are probabilistic. The returns-based beliefs equilibrium has both probabilistic actions and probabilistic beliefs. This is in contrast to other non-Nash equilibrium models which typically assume either probabilistic actions or probabilistic beliefs but not both. The third modification is to treat the actions/beliefs as *uncertain* rather than *random*. These terms are often used interchangeably in the literature. However, we make a distinction here between an uncertain player, who makes a conscious decision when faced with uncertain information about their opponent, and a random player, whose interpretation of the game changes subconsciously every time they encounter it. In the returns-based-beliefs model, when a given player plays a given game against a given opponent, the player forms the

[2] Kadane and Larkey (1982) in their seminal paper propose a decision analytic solution concept over the game theoretic solution concept. Also, see Bono and Wolpert (2009).

[3] Blavatskyy (2008) shows that the choice probabilities have a stochastic utility representation if they can be written as a non-decreasing function of the difference in the expected utilities of the actions. Studies have shown that the stochastic utility representation has properties that are similar to the Luce (1959) choice rule when the random variable has a double exponential distribution function (e.g., the cumulative distribution function of the logistic distribution) (Yellot 1977).

same beliefs and has the same actions. These beliefs and actions are probabilistic due to the player's uncertainty about the cooperative stance of the opponent and we encapsulate this uncertainty through the use of the Luce choice rule and the use of hyperpriors, which will be described in more detail below. This is different to the random actions/beliefs used in other equilibrium concepts, in which the interpretation is that every time a player plays a game, his actions or beliefs change probabilistically, and the "equilibrium" solution is an ensemble average. This is a subtle distinction, but there is also a mathematical difference — in the returns-based-belief model, we compute a player's best response to the average action, as opposed to the average of his best response to a particular action. We discuss the similarities and differences of the returns-based beliefs model with these other non-Nash equilibrium models below.

Our model has similarities with the Quantal Response Equilibrium (QRE) model proposed by McKelvey and Palfrey (1995) and the Boundedly Rational Nash Equilibrium (BRNE) model of Chen, Friedman and Thisse (1997). However, in the QRE and BRNE, the players only have probabilistic actions and not probabilistic beliefs. Additionally, in the QRE, the actions are random rather than uncertain — the equilibrium is an average of the best response to a game that includes errors in the payoffs. The QRE can be viewed as an application of stochastic choice theory to strategic games or as a generalization of the Nash equilibrium that allows noisy optimization (Haile, Hortacsu and Kosenok 2008). The QRE and its close variant the BRNE assumes that the decision maker might take a suboptimal action, and that the probability of doing so increases with the expected payoff of the action. Each decision maker follows the random choice interpretation of discrete choice theory under which each player selects a mixed strategy in order to achieve the best possible outcome given the randomness in the utilities of the player making the decision. In the BRNE and QRE models, the decision maker plays strategies proportional to the expected payoff with some error. The error the decision maker makes could be interpreted as either unmodeled costs of information processing (McKelvey and Palfrey 1995) or as unmodeled determinants of utility from any particular strategy (Chen, Friedman and Thisse 1997). We build on the QRE and BRNE models to develop our approach. Our model has some mathematical similarities to the QRE and BRNE. In particular, the 'returns-based beliefs' is the expected returns of a particular strategy, in proportion to the total expected returns of all strategies. However, the returns based beliefs approach differs significantly from the QRE and the

BRNE because our approach does not assume unmodeled costs of information processing or unmodeled determinants of utility from any particular strategy. We also do not assume that the decision maker is making errors or mistakes. Rather in the returns based beliefs model, individuals need to form subjective beliefs about each others possible plays. Since there is strategic uncertainty or ambiguity about these beliefs we invoke the concept of hyperpriors in the formation of such beliefs. In this respect our model differs both conceptually and mathematically from both the QRE and the BRNE[4].

Our model also has some similarities to the random belief equilibrium (RBE) proposed by Friedman and Mezzetti (2005). However, in the RBE, the players only have probabilistic beliefs and not probabilistic actions. Moreover, the beliefs are random, not uncertain — each time a game is played, the player forms a different belief about his opponent's strategy and the equilibrium is the ensemble average of the best-response over many realizations of the game. In the RBE, each player attempts to play a focal strategy which could be the Nash equilibrium strategy. However, each player faces some uncertainty as to the choice of the other player at equilibrium. They postulate that a player's belief about the mixed strategy choices of the other player are randomly drawn from some belief distribution that is defined over the mixed strategy set of the opponent and view this distribution as being dispersed around a central strategy profile. In the RBE each player chooses a best reply to her beliefs and the beliefs are statistically consistent in that the expected choice of each player coincides with the focus of the belief distributions of the other players about the first player's choice. The RBE assumes that each player chooses the pure strategy that maximizes their return based on their belief about the random draw of the other player from the belief distribution. The average of these pure strategy best responses then gives the mixed strategy profile of the players. In the returns based belief equilibrium, we similarly assume that each player faces uncertainty as to the choice of the other player in equilibrium. This uncertainty stems from the desire to want to cooperate in order to maximize the payoff for each player. Each player then needs to form a subjective probability about the other

[4] Our model based upon returns-based beliefs is also different to other alternative models that provide an explanation for non-Nash equilibrium outcomes such as the level-k models (Costa-Gomes and Crawford 2006) and cognitive hierarchy (CH) model (Camerer, Ho and Chong 2004). The level-k and CH models assume that decision makers are heterogenous in their level of sophistication with respect to their strategic thinking. The returns-based beliefs model differs from the level-k and the CH models in that the former does not assume heterogeneity in the levels of sophistication in thinking by decision makers.

player's strategy and probabilistically best responds to this belief. We assume that the subjective probability is consistent with certain axioms which gives the Luce probabilistic choice rule as the basis of the belief formation. Similar to the RBE we assume that the beliefs are statistically consistent. In the returns based beliefs equilibrium, each player forms subjective probabilities over the actions of the individual's opponent and chooses a mixed strategy profile over the actions based on the relative returns. Therefore, unlike the RBE, in the returns based beliefs approach the player's action is a mixed strategy profile which is averaged and then maximized. In the RBE model, every time a player plays a game against a given opponent, their beliefs change as they are chosen randomly from the belief distribution. In the returns based beliefs model beliefs are uncertain rather than random — a player makes a definite decision based on his belief distribution and would therefore always adopt the same mixed strategy playing a given game against a given opponent.

We first show how the returns based belief model without ambiguity is able to provide qualitative results that are consistent with empirical findings from the Prisoner's Dilemma. We then show that the returns based belief model with hyperpriors to incorporate ambiguity is able to better explain the empirical data from completely mixed 2×2 games from Selten and Chmura (2008) once loss aversion is included compared to other stationary concepts. Thus, by putting forward the concept of returns-based beliefs we contribute to the extensive literature which tries to reconcile game theoretic predictions and empirical experimental results (Holt and Roth 2004). The next section develops the returns based beliefs model. Section III applies the returns based beliefs model to Selten and Chmura's (2008) completely mixed 2×2 games. Section IV concludes.

II. RETURNS-BASED BELIEFS

Game theory models systematically human behavior when strategic interactions exist. In conventional game theory, the solution concept such as Nash equilibrium is critical in forming the basis for the prior distribution of beliefs that players hold. In determining the outcome of the game these prior beliefs held by the players are fulfilled in equilibrium. However, a player's actions are determined by her beliefs about other players which may depend upon their real-life contexts such as custom or history. For example, Harsanyi (1982) contended that normative game theory was not as helpful as 'an empirically supported psychological theory making probabilistic predictions about the strategies people are likely to use, . . . given the nature of the game and given their own psychological makeup' (Harsanyi 1982, p.122). This psychological makeup might be conditioned by the past experience of individuals' beliefs about an opponent's play. This is termed the 'subjective' or personal interpretation of probability. Subjective probability is the probability that a person assigns to a possible outcome, or some process based on his own judgment, the likelihood that the outcome will be obtained (DeGroot 1975, p. 4; Savage 1954). The implication is that the experiences of the individual might feed into the so-called perceptive and evaluational premises of the individual and influence thereby the subjective probabilities, which then influences the strategies chosen. This approach calls for a decision theoretic approach whereby players form subjective beliefs about the opponent in choosing an action (Roth and Schoumaker 1983). However, as pointed by Myerson (1991, p.114), 'A fundamental difficulty may make the decision-theoretic approach impossible to implement, however. To assess his subjective probability distribution over other players' strategies, player i may feel that he should try to imagine himself in their situations. When he does so, he may realize that the other players cannot determine their optimal strategies until they have assessed their subjective probability distributions over i's possible strategies. Thus, player i may realize that he cannot predict his opponents' behavior until he understands what an intelligent person would expect him rationally to do, which is, of course, the problem that he started with.' To resolve the issue, psychological motives could be used to understand behavior in game theoretic settings. Economists have suggested examining the process of cognitive reasoning (Rubinstein 2007). Research in psychology and economics shows that people have a bias towards wanting to cooperate (Farrell and Rabin 1996; Andreoni and Miller 1993). We follow this line of reasoning to propose a reasonable way of forming such subjective probabilities.

We argue that the willingness to cooperate might be influenced by past experience, generating 'strategic uncertainty' regarding the conjecture about the choice of the other player. We define strategic uncertainty as uncertainty concerning the actions and beliefs (and beliefs about the beliefs) of others (Brandenberger 1996). Researchers have argued that strategic uncertainty can arise even when all possible actions and returns are completely specified and are common knowledge (Ellsberg 1959; Van Huyck et al 1990). The rational decision-maker has to form beliefs about the strategy that the other decision-maker will use as a result of strategic uncertainty. As a consequence, players form their beliefs about the probabilities that other players play in order to determine in turn their best-response strategy. Hence, the best response strategy of one player is likely to be based upon the mixed strategy of the other player. The mixture is because of the uncertainty regarding the conjecture about the choice by the other players[5] (Brandenberger and Dekel 1987). This is succinctly summarized by Rabin that 'In psychological games, there can be a difference between interpreting mixed strategies literally as purposeful mixing by a player versus interpreting them as uncertainty by other players' (Rabin 1993, p.1286). Following this line of reasoning, a player who knows that the non-Nash equilibrium belief is held by the opponent could form a subjective assessment of the opponent's play by taking this belief into account (Basu 1990). For example, in the Prisoner's Dilemma a player might play the cooperate strategy if they expect the other player to play the cooperate strategy. Therefore, player 1 has a profitable deviation by playing cooperate with a positive probability when player 2 plays cooperate with a positive probability[6].

In order to illustrate our model we need some plausible set of assumptions about how agents form beliefs with respect to the probability of the opponent's strategy. We suggest that one way in which agents might do so is by basing their decisions on Luce's (1959) probabilistic choice model. In this approach, players form subjective beliefs and then act based upon the expected returns, given these beliefs, of a particular strategy, in proportion to the total expected returns of all strategies. We call this the returns-based beliefs model.

[5] We are not assuming that the opponent is using a randomized strategy. The mixture merely reflects the representation of player 1's belief about player 2. As Wilson (1986, p.47) points out, although it makes little difference to the mathematics, conceptually this distinction between randomization and subjective beliefs to explain the mixed strategies is a pertinent one. This interpretation is also sympathetic to Harsanyi's (1973) purification interpretation of mixed strategy where mixing represents uncertainty in a player's mind about how other players will choose their strategies, rather than deliberate randomization (Morris 2008).

[6] Conventionally, any mixed strategy will have a support in pure strategies. However, the pure strategy (cooperate) will get eliminated by the deletion of dominated strategies which suggests that the play of a mixed strategy based upon a support in pure strategies would not be apposite in this context. However, because of our argument that invokes subjective probabilities, all the strategies are played with positive probabilities which includes the cooperate strategy. Therefore, a mixed strategy can exist if one player experiences uncertainty with respect to his conjecture about the choice of the other player.

A. The model

In this section we develop the basic elements of a one-shot game and then introduce discrete choice theory based on subjective probabilities. We use below a particular variation of the one-shot game formulation of McKelvey and Palfrey (1995). Let (N, S, π) be a finite game. N denotes the set of players. Each player $i \in N$ has a set of pure strategies, S^i with elements s^i. The set of strategy profiles of all players other than i is $S^{-i} = \otimes_{j \neq i} S^j$ with elements, s^{-i}. The benefit a player i derives from playing a strategy profile $s^i \in S^i$ is $\pi^i_{s^i}; \pi^i = \{\pi^i_{s^i}\}_{s^i \in S^i}$. Each player knows who is in the game, N, and the strategy sets available to each other, $S^i \; \forall \; i \in N$. However, each player is uncertain of the belief structure held by the other player. We discuss below our approach to how players form a reasonable subjective probability belief structure.

In the following discussion, we assume that each decision maker follows the random choice interpretation of discrete choice theory under which each player selects a mixed strategy in order to achieve the best possible outcome given the uncertainty regarding the other player's choice[7]. As discussed earlier, driven by the desire to want to cooperate, for example in the Prisoner's Dilemma game, there is uncertainty regarding the conjecture about the choice of the other player. Hence, the player holds an opinion based on the subjective probability with respect to all of the unknown contingencies affecting his payoffs. In particular the player is assumed to have 'an opinion about the major contingency faced, namely what the opposing player is likely to do' (Kadane and Larkey 1982, p. 115). Kadane and Larkey (1982, p. 115) have expressed the implications of this line of thought very neatly as follows: '*If I think my opponent will choose strategy i $(i = 1, ...I)$ with probability p_i, I will choose any strategy j maximizing $\sum_{i=1}^{I} p_i u_{ij}$, where u_{ij} is the utility to me of the situation in which my opponent has chosen i and I have chosen j......the opponent's utilities are important only in that they affect my views $\{p_i\}$ of what my opponent may do....*'. Although any possible distribution of probabilities could be a possibility based upon the subjective method of forming them, we propose a reasonable subjective probability belief that the players might use when they do not know each other or their respective histories. We call this '*returns-based beliefs*'.

We believe it is reasonable to assume that the decision maker would assign probabilities

[7] We emphasize that although this is commonly called the random choice interpretation in the literature, we use it here to represent uncertainty.

based on the expected returns from playing the different strategies. Similarly, we assume that the opponent also assigns probabilities based on the opponent's expected returns given the probabilities of the focal decision maker. Following this line of reasoning, our analysis is based on a model for which the decision probabilities are proportional to the expected returns. We assume that agents form beliefs based upon the expected returns of a particular strategy over the total expected returns of all strategies, assuming the opponent plays all possible strategies.

Our proposed approach has both theoretical and empirical justification. First, for the theoretical justification we defer to Luce (1959) who showed using probability axioms that if the ratio of probabilities associated with any two decisions is independent of the payoff of any other decisions, then the choice probabilities for decision j of player i can be expressed as a ratio of the expected payoff for that decision over the total expected payoff for all decisions: $\Pi^i_j / \sum_k \Pi^i_k$, where Π^i_j is the expected returns associated with decision j. Second, Gul, Natenzon and Pesendorfer (2010) show that the Luce rule is the unique random choice rule that admits well-defined ranking of option sets. In particular, the authors show that the Luce rule is the unique random choice rule that admits a context-independent stochastic preference. Third, Blavatskyy(2008) develops a stochastic utility theorem by assuming that the primitive of choice is a choice probability and not a preference relation over actions. The paper shows that choice probabilities such as the Luce (1959) rule admit a stochastic utility representation if and only if they are complete, strongly transitive, continuous, independent of common consequences and interchangeable. Fourth, this method of arriving at decision probabilities has been supported by empirical work which provides empirical justification for our approach. In particular, empirical research for paired comparison data supports the Luce (1959) method of arriving at decision probabilities such that the probability of choosing x over y, $P(x, y) = v(x)/[v(x) + v(y)]$ where $v(x)$ and $v(y)$ are the scale values of choosing x and y respectively (Abelson and Bradley 1954).

In a game between multiple players, we assume that each player has a probability distribution over the choices available. This probability distribution, P^i, over the elements S^{-i} is defined such that $P^i(s^{-i})$ is the probability associated with $s^{-i} \in S^{-i}$. We operationalize our model by assuming an expected return framework, so that the expected payoff of the

jth pure strategy of player i, given P^i, is as follows[8]:

$$\Pi_j^i(P^i) = \sum_{s^{-i} \in S^{-i}} P^i(s^{-i}) \pi_j^i(s^{-i}) \tag{1}$$

where $\pi_j^i(s^{-i})$ is player i's payoff from choosing a pure strategy j when the other players choose s^{-i} and $P^i(s^{-i})$ is the belief held by player i about the probability the other players will choose s^{-i}. The decision probabilities over pure strategies for player i in turn follow the specification outlined above which is proportional to the expected returns as follows:

$$p_j^i = \frac{\Pi_j^i(P^i)}{\sum_{k=1}^m \Pi_k^i(P^i)} \tag{2}$$

This model admits a Nash-like equilibria in belief formation such that the belief probabilities match the decision probabilities for all players. This equilibrium in beliefs can be found by iterating between the expected payoff in equation (1) and the decision probabilities in equation (2). We discuss the epistemic condition for the returns based beliefs equilibrium in section II C.

The returns based belief model can be used to find equilibria for games between any number of players. However, for clarity, the games discussed in the rest of this paper will be games played between 2 players only. We will use u_{ij}^1 and u_{ij}^2 to denote the payoffs to player 1 and 2, respectively, if player 1 plays move i and player 2 plays move j. We will denote a mixed strategy of player 1 by $\mathbf{p} = \{p_i\}$, where p_i is the probability that player 1 chooses $s^i \in S^1$, and denote a mixed strategy of player 2 by $\mathbf{q} = \{q_j\}$, where q_j is the probability that player 2 chooses $s^j \in S^2$. The Luce rule, Eq. (2), defines a mapping $\mathcal{M}_{ij} : \sigma_j \to \sigma_i$ from mixed actions, σ_j, of player j to mixed actions, σ_i, of player i. This leads us to define the notion of a *returns-based belief (RBB) equilibrium* as

Definition A returns-based belief equilibrium is a pair of mixed actions, $(\mathbf{p},\mathbf{q})$, such that $\mathcal{M}_{12}(\mathbf{q}) = \mathbf{p}$ and $\mathcal{M}_{21}(\mathbf{p}) = \mathbf{q}$. In other words, this is a solution in which both players play the Luce-type response, and each player's belief about their opponent coincides with the actual strategy the opponent adopts.

[8] This formulation, similar to the BRNE with a logit distribution, is invariant to linear (i.e., scale) transformations, but not to changes in the payoff origin. On the other hand, the QRE with an extreme value distribution, accommodates negative payoffs and is invariant to changes in the payoff origin but not to linear transformations.

B. Properties of the returns-based belief equilibrium

We now state some properties of the RBB equilibrium. The following statements were proven in Velu, Iyer and Gair (2011), but we reproduce these proofs in a slightly extended form in the supplementary material provided with this paper.

Proposition 1 *Any game $(N, S, \mathbf{u})$, with non-negative payoffs $\mathbf{u}$, has a returns-based belief equilibrium.*

The requirement that the payoffs be non-negative is to ensure that the equilibrium solution has non-negative probabilities. However, we stress that this does not mean that RBB equilibria are only defined for games with positive payoffs. In an arbitrary game, the entries can be made strictly positive by replacing the payoffs by utilities in the payoff matrices. Even using payoffs, the potential problem of negative entries can be avoided by a simple modification in which we use a *probability-zeroing algorithm.* In this approach, we modify the Luce rule by setting $\Pi^i_j(\sigma) = 0$ when $\Sigma_m \mathbf{u}^i_{jk}\sigma_k < 0$ but $\Pi^i_j(\sigma) = \Sigma_m \mathbf{u}^i_{jk}\sigma_k$ otherwise. In this simple extension of the RBB model, it can be easily seen that the existence proof carries over to all games and so an RBB equilibrium always exists. The uniqueness of the RBB equilibrium in the case of two-player games (proposition 3) does rely on the payoffs being positive and therefore multiple RBB equilibria may exist in this extended model.

The proof of proposition 1 is presented in the context of games between two players, but it can be straightforwardly generalized to games involving an arbitrary number of players. However, in the case of two-player games the RBB equilibrium has some nice properties by virtue of the following proposition.

Proposition 2 *In games between two-players, the returns-based belief equilibria are given by eigenvectors of the matrix $\mathbf{u^1}\,(\mathbf{u^2})^{\mathbf{T}}$, where T denotes the transpose, $\mathbf{u^1} = \{u^1_{ij}\}$, $\mathbf{u^2} = \{u^2_{ij}\}$ and u^n_{ij} is the payoff to player n when player 1 chooses move i and player 2 chooses move j.*

This proposition leads to a uniqueness theorem for games with positive payoffs.

Proposition 3 *In a game between two players in which all the payoffs to both players are positive, i.e., $u^1_{ij} > 0$, $u^2_{ij} > 0\ \forall\ i, j$, the returns-based equilibrium is unique.*

If we relax the assumption of strictly positive payoffs and allow the elements in the payoff matrices to be zero, i.e., now $u^1_{ij} \geq 0$, $u^2_{ij} \geq 0 \ \forall \ i, j$, then RBB equilibrium solutions can lie on the boundary of the space of probability vectors, i.e., we can have $v^i = 0$ for some i. However, the argument used to prove proposition 3 can still be used to prove that *there is at most one interior RBB equilibrium*, i.e., at most one RBB equilibrium that does not lie on the boundary. However, multiple boundary RBB equilibria may exist in addition.

It would be unreasonable to suppose that players in a game would be computing eigenstates of matrices in order to decide on their best move. However, the RBB equilibrium solution can also be derived iteratively. If a player is returns-based, i.e., plays the Luce type response in proportion to the expected return, and believes that his opponent is also returns-based, then if he began with a guess σ^2_0 (we describe in the next section how such a prior might be formed) of his opponent's mixed strategy, he would play $\sigma^1_0 \propto \mathbf{u}^1\sigma^2_0$. From his belief that his opponent is returns-based, the player is led to the belief that his opponent will play an alternative strategy $\sigma^2_1 \propto (\mathbf{u}^2)^T\sigma^1_0$, so he can update his guess and repeat, iterating until he converges to a solution.

Our initial definition of the RBB equilibrium assumed that beliefs and strategies coincided. However, this iterative convergence to the RBB equilibrium suggests an alternative, equivalent, definition for the equilibrium

Definition A returns-based belief equilibrium is a solution in which each player plays the Luce-type response, and believes that their opponent will play the Luce-type response to their strategy, i.e., each player is "returns-based" and believes his opponent is "returns-based".

Hence the equilibrium can also be thought of as an equilibrium in beliefs. This is similar to Binmore's (2009, p.135) 'subjective probabilities whereby beliefs rather than strategies are treated as primary'. Camerer (2003, p. 150) makes a similar point that mixed strategy equilibrium can be seen as an equilibrium in beliefs. The returns-based beliefs approach is different from Nash equilibrium because players respond to their beliefs by placing probability on strategies in proportion to their expected payoff. The equivalence of these two definitions of the RBB equilibrium is demonstrated by the following proposition.

Proposition 4 *In a game between two players in which all the payoffs to both players are positive, the iterative algorithm always converges to the unique returns based belief equi-*

librium. If the payoffs are non-negative, the iterative algorithm converges to the unique non-boundary RBB equilibrium if it exists.

C. An epistemic characterization of the returns-based beliefs

The returns based equilibrium is an equilibrium in both beliefs and in actions. In the RBB equilibrium, conjectures coincide with actions: what players do is the same as what the other player believe they will do. Hence, in that sense, the conjectures as well as the actions are in equilibrium. In this section we compare the RBB to the Nash equilibrium and other equilibrium concepts in terms of the epistemic conditions for equilibrium.

Traditional solution concepts in game theory such as the Nash equilibrium assume common knowledge of rationality or at least a high degree of mutual knowledge of rationality. Aumann and Brandenburger (1995) show that for a two player game mutual knowledge of rationality is sufficient for Nash equilibrium. Mutual knowledge implies that player 1 only needs to know what player 2 believes about his strategy (which is rational in the sense of maximizing returns) and vice versa for player 2. Therefore, in a two player game there is no need for higher orders of beliefs for the outcome to be a Nash equilibrium.

We can apply the same reasoning of mutual knowledge to the RBB equilibrium. In a two player game, for the players to be in an RBB equilibrium we need each player to follow the Luce rule (players identify with rationality based on the Luce rule), to believe that their opponent is Luce type and that the strategy that they believe their opponent thinks they will adopt coincides with the actual strategy they adopt. However, the mutual knowledge equilibrium implies that the players do not have to know they are in the RBB equilibrium, but they do have to be in it "accidentally". The question which the mutual knowledge formulation does not address is how the players reach their belief about what strategy their opponent assumes about them. Presumably, for player 1 to believe that player 2 thinks that he, player 1, will play the RBB equilibrium strategy, then he must have gone through the iterative process in his head to reach that conclusion. So, whereas mathematically the different orders of beliefs are decoupled and we only need the mutual knowledge condition to hold, in practice the first order state of beliefs would be a product of iteration on higher order beliefs. In short, the mutual knowledge equilibrium requires the players to be in a particular belief state but does not explain how they reached that state or where their priors came from. The non-Nash equilibrium models discussed earlier such as the QRE, BRNE or RBE do not provide a theory of where the priors come from. This issue is neatly summarized by Chen, Friedman and Thisse (1997, p. 38-39) in their discussion of the BRNE: *Where do the choice probabilities come from? Admittedly, this is not clear; the probability of choosing a particular pure strategy depends on the subconscious utility attached to that strategy as compared with the subconscious utility attached to the other pure strategies. Meanwhile, the subconscious utility of a strategy cannot be computed without knowing the mixed strategies of the other players.* On the other hand, since the Luce rule comes from a set of axioms on how an individual would choose between choices in an uncertain environment (Blavatskyy 2008; Gul, Natenzon and Pesendorfer 2010, Luce 1959), the RBB provides a theory of the formulation of priors[9].

D. Characterizing uncertainty in beliefs

The returns-based belief equilibrium described above assumes that both players use the Luce (1959) choice rule and that player beliefs and strategies coincide. We can extend the model by allowing beliefs and strategies to differ, since there might remain some ambiguity as to the cooperative stance of the other player and hence, their disposition to form beliefs in such a manner. We will use a superscript 'b' to denote belief probabilities, i.e., $\mathbf{q}^b$ is the mixed strategy that player 1 believes player 2 will play and vice versa for $\mathbf{p}^b$. We will continue to assume that the way players act on their beliefs is based on the Luce expected returns approach, so that $\mathbf{p} = \mathcal{M}_{12}(\mathbf{q}^b)$ and $\mathbf{q} = \mathcal{M}_{21}(\mathbf{p}^b)$. We will denote the equilibrium solution in which $\mathbf{p} = \mathbf{p}^b$ and $\mathbf{q} = \mathbf{q}^b$ by a superscript 'rbbe', i.e., $\mathbf{p}^{\text{rbbe}}$, $\mathbf{q}^{\text{rbbe}}$.

We can capture the ambiguity in beliefs via the concept of hyperpriors that are formed over the beliefs developed via the expected returns approach as outlined above. Following this interpretation, the belief probability will arise as an integral over player 1's belief

[9] When there is an inter-linked belief structure whereby player 1 plays the Luce rule because he thinks player 2 beliefs that player 1 is a Luce type and vice versa (whereby player 2 plays the Luce rule because he thinks that player 1 thinks that player 2 is Luce type), there needs to be common knowledge of beliefs for the RBB equilibrium. The higher order belief structure requires a basis upon which the players derive their priors.

distribution, $P(\mathbf{q})$, for player 2

$$q_i^{\mathrm{b}} = \int q_i P(\mathbf{q}) \mathrm{d}\mathbf{q} = q_i^{\mathrm{rbbe}} \int \left[\frac{q_i}{q_i^{\mathrm{rbbe}}} P(\mathbf{q}) \right] \mathrm{d}\mathbf{q} = q_i^{\mathrm{rbbe}} X_i \tag{3}$$

where the last equality defines X_i from the belief distribution[10]. Writing the moments of the belief probability distribution in this way is an attempt to split the belief probability into a game dependent part, the equilibrium solution q_i^{rbbe}, and a player dependent part, X_i. We expect players who believe their opponent to be cooperative to play close to the equilibrium solution, and therefore $X_i \approx 1$. The Luce (1959) expected returns model provides a mapping between beliefs and actions, which allows the determination of these X_i parameters from experimental data. This will be explored further in Section III.

In standard statistical language, the strategy $\mathbf{q}^b$ is player 1's prior belief on the pure strategy that player 2 will play. The individual components of $\mathbf{q}$, which characterize the prior distribution, are hyperparameters. Player 1 may not be certain about the mixed strategy that player 2 will adopt and so he may have a distribution over the possible mixed strategies of player 2 and hence over the hyperparameters. Such a distribution is termed a hyperprior and is denoted by $P(\mathbf{q})$ in equation (3). In a theoretical approach, one could prescribe a hyperprior and derive the corresponding X_i's from equation (3). However, the consequences of the hyperprior are entirely encoded in the mixed belief strategy that the player forms, and hence in the X_i's. Thus, when presented with experimental data it is more practical to work with the measured q_i^b's or X_i's that can be determined from the observed strategies through the Luce (1959) rule.

We first examine the returns based belief model without ambiguity to provide qualitative results that are consistent with empirical results of the Prisoner's Dilemma. In particular, due to the limitation of the data for the Prisoner's Dilemma game we will only consider the returns-based-belief equilibrium solution, (p^{rbbe}, q^{rbbe}). We then go on to examine the empirical evidence from the Selten and Chmura (2008) data in section III B with the returns based belief model incorporating ambiguity using hyperpriors as described here[11].

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[10] Note that $\sum X_j q_j^{\mathrm{rbbe}} = 1$ since $\sum q_j = 1$.

[11] We note that deviations from the RBB equilibrium solution could also arise if a player makes an error in computation when evaluating the values associated with each move, and these values are ultimately used to determine their choice probabilities. In this case, we can use Y_i to represent the fractional error that player 1 makes when assessing the value of move i, so that he assigns a value $Y_i v_i^{\mathrm{rbbe}}$, where v_i^{rbbe} is

		Player 2	
		Cooperate	Defect
Player 1	Cooperate	(4, 4)	(0.5, 6)
	Defect	(6, 0.5)	(1, 1)

FIG. 1: Payoffs for the Prisoner's Dilemma game.

E. The Prisoner's Dilemma

We consider the Prisoner's Dilemma game as illustrated in Figure 1 where two agents have to decide whether to cooperate or to defect. Let us call the agents player 1 and player 2 respectively. If both cooperate they both get a payoff of 4. However, both player 1 and player 2 could be better off by playing defect regardless of what the other player does. Playing defect is the dominant strategy for both players and is the only Nash equilibrium of the Prisoner's Dilemma as there is no incentive for any of the players to change their strategies. This is clearly less than the Pareto optimum of obtaining a payoff of 4 each by both cooperating. Yet, empirical testing of the one-shot Prisoner's Dilemma in laboratory experiments has shown that people are prone to play the cooperative strategies far more often than the Nash equilibrium might lead us to predict. These studies have also demonstrated that as the benefits from cooperation increase, players are more likely to cooperate, and that as the loss from not cooperating increases, the likelihood of cooperation increases as well (Sally 1995).

Scholars have emphasized that cooperation in a one-shot Prisoner Dilemma game is an important finding of experimental research that needs to be further understood (Janssen

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the value of that move in the RBB equilibrium solution. As a result of these errors, player 1 will play a strategy with the belief probabilities

$$q_i^{\mathrm{b}} = \frac{Y_i v_i^{\mathrm{rbbe}}}{\sum_{j=1}^{N} Y_j v_j^{\mathrm{rbbe}}} = \frac{Y_i q_i^{\mathrm{rbbe}}}{\sum_{j=1}^{N} Y_j q_j^{\mathrm{rbbe}}} = X_i q_i^{\mathrm{rbbe}} \tag{4}$$

where $\sum X_i q_j^{\mathrm{rbbe}} = 1$. Mathematically, this interpretation is therefore identical to the ambiguity in beliefs characterized by the hyperprior.

2008). Economists have proposed various explanations for these experimental findings: for example, altruistic punishment among genetically unrelated people when the gains from reputation are small or absent (Fehr and Gachter 2002); the ability to recognize untrustworthy opponents (Janssen 2008); and the incorporation of notions of fairness into game theory through which people help others that help them and hurt others that hurt them (Rabin 1993). To all of these explanations, we add an alternative: a new explanation for cooperation in a one-shot Prisoner's Dilemma game that is based upon subjective probabilities and returns based beliefs.

In the case of the Prisoner's Dilemma, ideally, the agents would like to cooperate by coordinating their actions on the joint strategies that will maximize their returns, which is (4, 4). Intrinsically, each player knows the benefits of cooperation and hence he or she may actually play the cooperative strategy with positive probabilities, which coordinates with the other player. For example, research has shown that human beings are prone to cooperative behavior based on reciprocity (Axelrod 1984; Farrell and Rabin 1996). Therefore, the history of human interactions is likely to influence the general disposition of players to want to cooperate[12]. We need to factor this cooperative bias into our decision-making framework in order to predict how players should behave in a competitive situation where cooperation is possible and could produce better outcomes[13].

In the Prisoner's Dilemma, $S^i = \{C, D\}$ and $N = \{1, 2\}$. We could begin with any arbitrary beliefs and through a process of iteration each player updates their beliefs using the expected payoff in equation (1) and the decision probabilities in equation (2). Conducting this iterative process shows that these probabilities do actually converge after about three to four iterations, which is consistent with the results of Proposition 4. Since the players are symmetric, it is not unreasonable to assume without any further information about history or preferences that they would have the same subjective beliefs about each other.

[12] Farrell and Rabin (1996) have argued that even with communication there would be 'cheap talk' and hence the communication is not credible. However, Sally (1995) shows that communication does result in an increase in cooperation among players.

[13] To add to this line of reasoning, psychologists have argued that cooperation may be prompted by altruism, by the desire to conform to social norms, or by adhering to the dictates of one's conscience (Farrell and Rabin 1996). In addition, economists have shown that 'people's natural tendency to cooperate' is an important trait that subjects bring to experimental situations from the outside (Andreoni and Miller 1993, p. 571).

The probabilities converge for both players to 0.387 for the cooperate strategy and 0.613 for the defect strategy. Therefore, the 'returns based beliefs' show that the players will play the cooperate strategies with positive probabilities which is in line with empirical evidence. In addition the empirical evidence shows that as the benefits from cooperation increases the players are more likely to cooperate. Sally (1995) showed using a metanalysis of over 100 studies that 'The one major consistency with rational self interest is that the temptation to defect decreases the level of cooperation' (Sally 1995, p.75). Another way of looking at this is that as the opportunity to increase one's reward by defecting from unanimous cooperation decreases, then the likelihood of cooperation decreases. In addition, the analysis shows that as the loss from not cooperating increases, the likelihood of cooperation increases too. Sally (1995) proposed a method of calculating the Temptation Index and Loss Index, which for 2 players reduce to

$$\text{Loss} = \frac{C(2) - D(0)}{C(2)} \tag{5}$$

$$\text{Temptation} = \frac{D(1) - C(2)}{C(2)} \tag{6}$$

where $C(x)/D(x)$ are the payoffs to cooperators/defectors when there are x cooperators. For the game in Figure 1, the value of the indices are Loss = (4-1)/4=3/4 (75%) and Temptation = (6-4)/4=1/2 (50%). As one allows the benefit from cooperation from the (Cooperate, Cooperate) payoff to increase from 4 to 5.8, the loss index increases from 75.0% to 82.8% which in turn increases the probabilities of cooperation in the returns based equilibrium from 0.39 to 0.47. In addition, as the payoff from (Defect, Cooperate) is increased from 6 to 7.8, the temptation index increases from 50.0% to 85.0% which in turn decreases the probability of cooperation in the returns based equilibrium from 0.39 to 0.35. These results are consistent with empirical evidence from the Prisoner's Dilemma game, whereby: (1) there is an inverse relationship between cooperation and the temptation to defect and (2) there is a positive relationship between cooperation and the gains from cooperation.

F. The Prisoner's Dilemma with loss aversion

The argument we have made above for a player to, rationally, choose to play the dominated cooperate strategy in the Prisoner's Dilemma is that the (C,C) solution has a better payoff than the natural Nash solution of (D,D). The idea is that a player is willing to cooperate based on the belief that the other is willing to cooperate in the hope that this tendency will lead to a better payoff overall. Two general features that we would like a model of this decision process to include would be 1) if the (C,C) strategy does not pay better than the (D,D) strategy, then there should be zero probability that the player would choose C; 2) a player who is averse to risk should be less inclined to play C, since there is a risk that his opponent will defect, leading to a worse payoff for himself. The model that we have adopted, in which the players play mixed strategies according to the Luce decision rule, does indeed exhibit both of these features, if the PD payoff matrix is transformed to include loss aversion (see, for example, Selten and Chmura (2008) and Brunner et al. (2010)).

Player 1 can compute his guaranteed payoff by finding the max-min of the elements in his payoff matrix, u^1, i.e., for each possible move, i, he computes his minimal payoff, $min_j u^1_{ij}$, over possible strategies of his opponent, and then finds the maximum, $max_i(min_j u^1_{ij})$. If the maximum occurs for $i = I$, for which the minimum is at $j = J$, then by playing move I the player could guarantee a payoff at least as big as u^1_{IJ}. The matrix can then be transformed by subtracting u^1_{IJ} from each element of the matrix. The transformed payoff matrices represent the payoffs relative to the guarantee. The RBB approach naturally places unfavorable weight on negative payoffs, so this transformation already encodes loss aversion. However, the standard approach is to multiply negative entries in the transformed matrix by a multiplier, λ. This multiplier is a measure of the degree of loss aversion of the player.

An analysis of the PD game including loss aversion demonstrates that it has precisely the features we wanted (as noted in (1) and (2) above). This analysis is provided in the supplementary material to this paper. A solution with some cooperation exists if there is an incentive to mutual cooperation, i.e., the payoff from defect-defect is worse than that from cooperate-cooperate. As that incentive goes away, so does the tendency to cooperate. In addition, we see that the more loss averse a player is, the less inclined they will be to cooperate and sufficiently loss-averse players will always choose the Nash equilibrium. These nice features illustrate why the Luce choice rule provides a good model for cooperative behavior.

We note that when loss aversion is included, it will in general introduce negative entries into the transformed payoff matrices. The existence and uniqueness proofs present above assumed non-negative payoffs and so these no longer apply. When the loss aversion multiplier, λ, is set to 0, the entries are non-negative and the results proven above apply, but as the loss aversion multiplier is increased, different things can happen depending on the exact specification of the game. In the Prisoner's Dilemma example described above, two RBB equilibria for $0 < \lambda < (X - Y)^2/(4P(X - Y + R))$, but there are no RBB equilibria for $\lambda > (X - Y)^2/(4P(X - Y + R))$. However, for 2×2 constant sum games, a unique RBB equilibrium exists for all choices of the loss aversion parameter, which we discuss in the next section. A proof is provided in the supplementary material to this paper. Nonetheless, for any specification of the game and any value of the loss aversion multiplier, the probability-zeroing algorithm described above will always have a fixed-point and provides a natural extension of the RBB equilibrium concept.

III. EXPERIMENTAL 2×2 GAMES

In order to test the applicability of the RBB model in an empirical setting we will look at data from Selten and Chmura (2008). They presented empirical results for 12 games, each played between two players and with two possible moves for each player. The games are asymmetric, games 1-6 are constant sum games, but games 7-12 are non-constant sum. The payoffs for all twelve games are shown in Figure 2. The constant sum and non-constant sum games form pairs (e.g., Game 1 with Game 7, Game 2 with Game 8 and so on). As noted by Selten and Chmura (2008, p. 950), 'the non-constant sum game in a pair is derived from the constant sum game in the pair by adding the same constant to player 1's payoff in the column for R and player 2's payoff in the row for U'.

A. Loss aversion

Before analyzing the data from the Selten games, we first transform the pay-off matrices to include loss-aversion (consistent with the Impulse Balance Equilibrium in Selten and Chmura 2008, p. 947). It was demonstrated by Brunner et. al (2010) that the explanatory power of the various models compared by Selten and Chmura (2008) improves significantly once loss aversion is included before computing the equilibrium solutions. We include loss aversion by subtracting the guaranteed payoff from each entry in the matrix, as described in Section II F and then multiply negative entries in the transformed matrix by a multiplier, λ. The conventional approach is to take $\lambda = 2$ (Selten and Chmura 2008), but we will show

Game 1

(10,8)	(0,18)
(9,9)	(10,8)

Game 7

(10,12)	(4,22)
(9,9)	(14,8)

Game 2

(9,4)	(0,13)
(6,7)	(8,5)

Game 8

(9,7)	(3,16)
(6,7)	(11,5)

Game 3

(8,6)	(0,14)
(7,7)	(10,4)

Game 9

(8,9)	(3,17)
(7,7)	(13,4)

Game 4

(7,4)	(0,11)
(5,6)	(9,2)

Game 10

(7,6)	(2,13)
(5,6)	(11,2)

Game 5

(7,2)	(0,9)
(4,5)	(8,1)

Game 11

(7,4)	(2,11)
(4,5)	(10,1)

Game 6

(7,1)	(1,7)
(3,5)	(8,0)

Game 12

(7,3)	(3,9)
(3,5)	(10,0)

FIG. 2: Pay-offs for the 12 games in Selten and Chmura (2008). In each cell an entry (x, y) indicates a payoff of x for the row player and y for the column player.

results for both $\lambda = 1$ and $\lambda = 2$ in the following.

After carrying out this transformation, some of the initially positive pay-offs become negative. Our previous results no longer guarantee the existence of the RBB equilibrium solution under those circumstances. However, it is possible to prove for 2×2, constant-sum games, that an RBB equilibrium exists for any choice of the loss multiplier used when transforming the matrices to include loss-aversion. This proof is provided in the supplementary material to this paper. In the non-constant sum case it is more difficult to prove generic existence of the RBB equilibrium solutions, although such solutions do exist for all the games considered by Selten and Chmura (2008). Two of the games, numbers 8 and 10, have a singular behavior that arises from the equality of pay-offs for some moves. The derivation of the RBB equilibrium solution is more subtle in those cases but still straightforward. The method is described in the supplementary material to this paper.

B. Ambiguity in beliefs

In this section we will explain the empirical observations of the completely mixed 2×2 games using the RBB model. The RBB model incorporates ambiguity using a hyperprior on the belief probability, as described in equation (3). The experimental data can be used to determine the properties of this hyperprior, which we characterize in terms of the X_i parameters. These are determined from the belief probabilities held by the players, which we must compute from the observational data. Under the Luce (1959) rule, the probability of move i for player 1 is just $p_i \propto u^1_{ij} q_j$, where q_j is the player's belief about player 2's strategy. In the RBB equilibrium, q_j is determined from p_i via a similar equation. However, if player 1 is observed to play the strategy $p_i^{\rm obs}$, then this equation can be inverted to give $q_j^{\rm b} \propto (u^1)^{-1}_{jk} p_k^{\rm obs}$, which is the belief that player 1 must have had about the strategy of player 2 in order to lead them to play as they did. For any results observed in an experiment, the belief probabilities $q_j^{\rm b}$ that would lead to the observed behavior, $p_i^{\rm obs}$ can be computed in this way. If the person is playing a pure RBB equilibrium strategy, then $q_j^{\rm b} = q_j^{\rm rbbe}$, but the ambiguity characterized by the hyperprior would lead to some deviation.

For two-move games, there is only one probability, which we denote $q = q_1$, as $q_2 = 1 - q$. In Figure 3 we show $q^{\rm b}$ versus $q^{\rm rbbe}$ and $p^{\rm b}$ versus $p^{\rm rbbe}$ for all of the data in the Selten and Chmura games. The two plots correspond to transformed games with $\lambda = 1$ and $\lambda = 2$ respectively. We see that there is a tight correlation between the beliefs that players hold and the beliefs predicted in the RBB equilibrium, which reinforces the idea that the RBB equilibrium is a good model for describing choice behavior in games. However, there are some deviations as we expected.

As described above, deviations will arise because the player is ambiguous as to the cooperative stance of the other player and therefore the likelihood that their opponent will play the RBB equilibrium solution. In order to incorporate this ambiguity we examine the RBB model with hyperpriors. As mentioned earlier, the belief distribution, $P(\mathbf{q})$, is a *hyperprior* as it is a distribution for the hyperparameters, $\{q_j\}$, that characterize player 1's prior be-

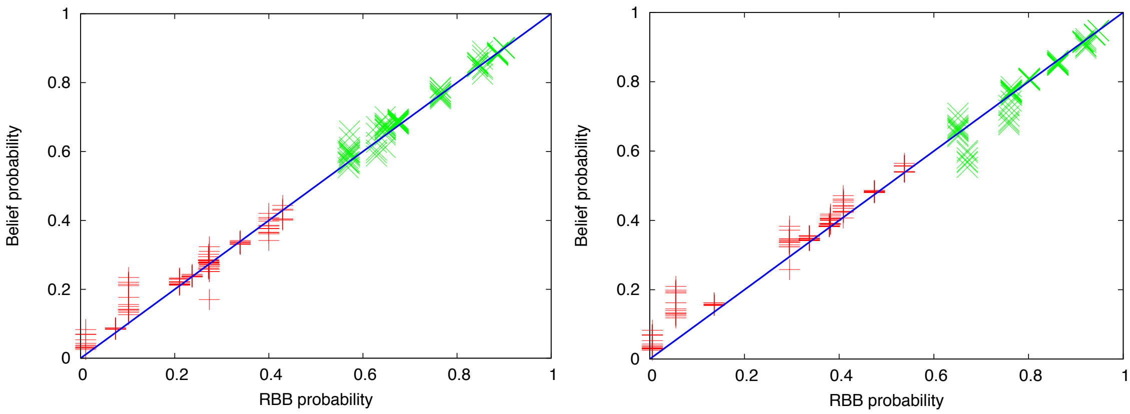

FIG. 3: Belief probability (y axis) versus RBB equilibrium probability (x axis) for all data in Selten and Chmura. The red plusses are values for p, the probability for the row player, while the green crosses are for q, the probability of the column player. The line indicates $q^{\mathrm{b}} = q^{\mathrm{rbbe}}$ to guide the eye. The left panel uses a multiplier $\lambda = 1$ to transform the game, while the right panel uses $\lambda = 2$.

lief, $\mathbf{q}$, for the action of player 2. A particular choice of hyperprior will lead to particular $\{q_i^{\mathrm{b}}\}$'s, but the observational consequence of the hyperprior is entirely encoded in the X_i parameters. These represent the fractional change from the RBB equilibrium solution. The deviation could also be characterized by the values of the observed belief probabilities, q_i^{b}, directly, but using the X_i parameters allows us to factorise the belief into a game-dependent part, $\mathbf{q}^{\mathrm{rbbe}}$, and a part that is a property of the players in the game, $\{X_i\}$. This is not the only way in which such a factorization could be accomplished, but it provides a good description of the observed data and it has a simple interpretation as a fractional modification of the belief probability. The $\{X_i\}$'s can be computed for any observed data, but to test whether they are indeed game independent for a particular set of players would require new experiments[14].

In a given data set, the probability constraint $\sum q_i^{\mathrm{b}} = \sum q_i^{\mathrm{rbbe}} = 1$ means that the X_i's are not all independent. Alternatively, we can define $r_i = X_i/X_N$ for $i = 1, \cdots N-1$ and then the r_i's can independently take any values in $[0, \infty)$. In the case of the Selten and Chmura data, each game has only two moves and therefore one independent probability and one r_i, $r_1 = X_1/X_2$. This quantity can be computed from the observed probability q^{b} and

[14] This is a common problem to any analysis of the belief distribution, since it is reasonable to suppose that a player might formulate his beliefs about his opponent in a way that depends on the properties of the game.

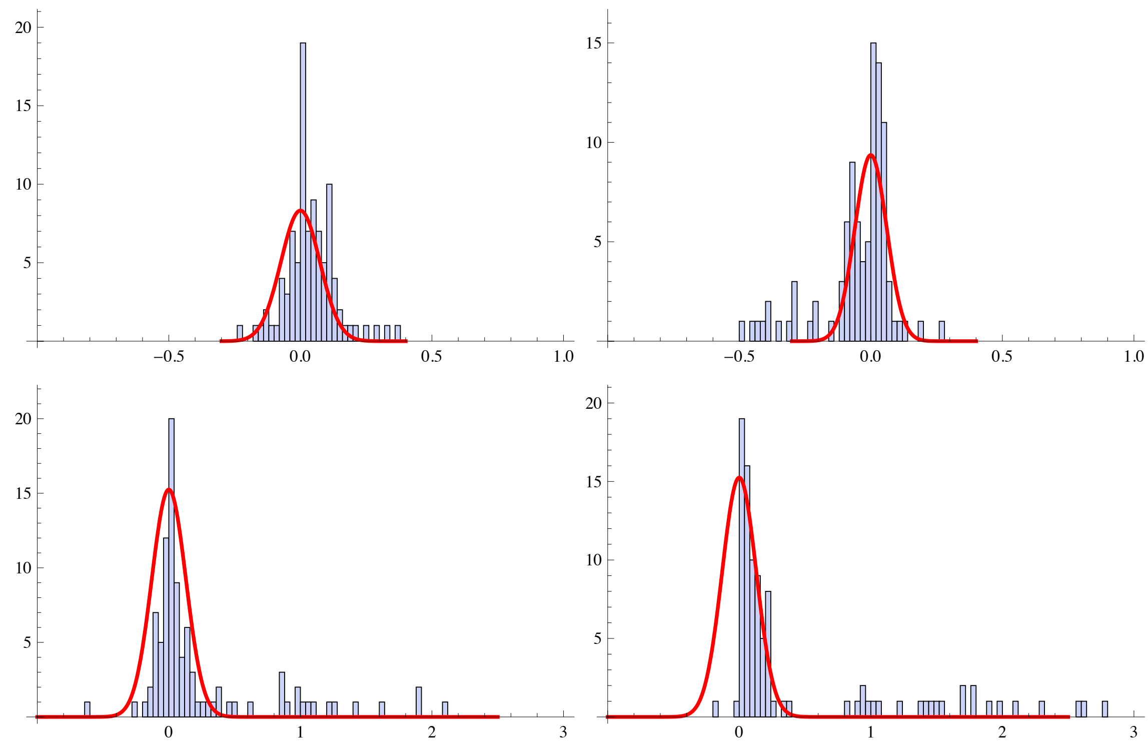

FIG. 4: Distribution of the parameter r_1 for player 1's belief probability, q^{b}, (upper panels) and player 2's belief probability, p^{b}, (lower panels). The solid lines are the best-fit zero-mean log-normal distributions to the data. The lefthand panels use a multiplier $\lambda = 1$ to transform the games, while the righthand panels use $\lambda = 2$.

the RBB equilibrium probability q^{rbbe} as

$$r_1 = \frac{q^{\mathrm{b}}(1 - q^{\mathrm{rbbe}})}{q^{\mathrm{rbbe}}(1 - q^{\mathrm{b}})}. \tag{7}$$

The r_1 parameter encodes information about the players playing the game. We want to characterize the distribution of values this parameter can take using the experimental data. The two moves for player 1 are treated asymmetrically in r_1, as we have chosen move 1 to be in the numerator. As the moves could be ordered arbitrarily, using $r_2 = X_2/X_1 = 1/r_1$ is equally valid. If r_1 is an intrinsic property of people, we would therefore expect its distribution to be inversion symmetric, i.e., the distribution of $1/r_1$ should resemble that of r_1. A simple distribution that has this property is a log-normal distribution with zero mean. In Figure 4 we show the distribution of r_1 for the Selten and Chmura data, along with the best-fit zero-mean log-normal distributions. The left and right hand panels are for loss aversion multipliers $\lambda = 1$ and $\lambda = 2$ respectively.

The fits in Figure 4 reproduce the data quite well, which lends support to the picture we have developed, namely that the players are formulating their beliefs according to the RBB equilibrium strategy, but with fractional modifications that arise from the player's beliefs about the cooperability of their opponent. Characterized in this way, the beliefs seem to be consistent with being drawn from a zero-mean log-normal distribution. There are a few things to note from the figure. Firstly, we have treated player 1 and player 2 independently by analyzing the q^{b} and p^{b} results separately. The games considered do not treat player 1 and player 2 equivalently in terms of the payoffs. Nonetheless, the widths of the best-fit distributions are quite similar in both cases, which supports the notion that the X_i's are an intrinsic characteristic of the players rather than the game. Secondly, we note that there are some outliers in the distribution for p^{b}. These all come from the separate replicates of game 1. Game 1 has a very small value for p^{rbbe} — the row player is unlikely to play the upper move due to the potential penalty involved. This has two consequences — (i) the statistical errors in measurements of this number are proportionally larger; (ii) statistical errors are magnified, resulting in larger errors in the X_i's. For this reason, we treated these points as outliers when deriving the best-fit distributions. Finally, we see that the fits seem to be somewhat better for $\lambda = 1$ than for $\lambda = 2$. The choice of this parameter is to some extent arbitrary and $\lambda = 2$ is chosen by convention, but this result might indicate that this set of players were not particularly loss averse.

C. Comparison to other equilibrium concepts

In Selten and Chmura (2008), the observations described here were compared to the predictions of several equilibrium concepts — the Nash equilibrium, the QRE, the Action Sampling equilibrium, the Payoff Sampling equilibrium and the Impulse Balance equilibrium. In the original analysis, it was found that the Impulse Balance equilibrium was a much better predictor of behavior. However, the impulse balance approach naturally incorporates loss-aversion and it was subsequently demonstrated by Brunner et al. (2010) that if the analysis was repeated on the transformed game including loss aversion with $\lambda = 2$, the equilibrium concepts other than Nash are able to explain the data as well as the impulse balance results. To compare the RBB equilibrium predictions to those of these other concepts, it is natural to use the same deviation parameter, r_i, introduced above. The alternative equilibrium concepts are not formulated in terms of beliefs about a player's opponent so in this context the X_i parameters are most readily interpreted as the fractional errors made by the player in the computation of the equilibrium solution. The RBB equilibrium solution represents an equilibrium in beliefs and therefore we compare the belief probabilities to the RBB equilibrium solution, rather than the move probabilities. This approach is natural in the RBB model because of the assumptions made about the players' decision process, but the other equilibrium concepts are not formulated in terms of beliefs about the opponent [15]. For the other models, we must therefore compare the move probabilities for each player to the predictions of the equilibrium concept by defining

$$r_1 = \frac{q^{\text{obs}}(1 - q^{\text{pred}})}{q^{\text{pred}}(1 - q^{\text{obs}})} \tag{8}$$

where q^{obs} is the observed probability that player 2 plays move left and q^{pred} is the corresponding probability predicted by the particular equilibrium concept under consideration.

In Figure 5 we show the distribution of the logarithm of this parameter, r_1, for each of the equilibrium concepts listed above. The bottom right panel shows the corresponding distribution for the RBB model for comparison. We used a loss aversion multiplier $\lambda = 2$ for consistency with Brunner et al. (2010). The shape of these distributions is broadly consistent with a normal distribution in all cases so the treatment of the deviations from the model described here could be applied to any of the other equilibrium concepts. The width of the distribution reflects the amount of departure from the equilibrium solution seen in the data. We see that the Nash prediction is generally worse than the other predictions, which is consistent with previous results in Selten and Chmura (2008) and Brunner et al. (2010). It is also clear that the RBB equilibrium solution is generally better than the other concepts, showing a much narrower distribution in the deviation parameter.

For a more direct comparison to Selten and Chmura (2008) and Brunner et al. (2010) we can compare goodness of fits for these various equilibrium concepts. We first compute the squared error in the equilibrium prediction for each data point, i.e., $(p_X - p_{obs})^2 + (q_X - q_{obs})^2$, where (p_X, q_X) are the equilibrium model predictions for "up" and "left". For the RBB model we compare the belief probabilities to the equilibrium predictions, as above, while for the other concepts we compare the move probabilities to the predictions. For consistency with

[15] Note that it is possible to reformulate some of the other equilibrium concepts, for instance the QRE and Nash solutions, in terms of a hyperprior on the belief probabilities of one player about the other. Such models are worth further investigation.

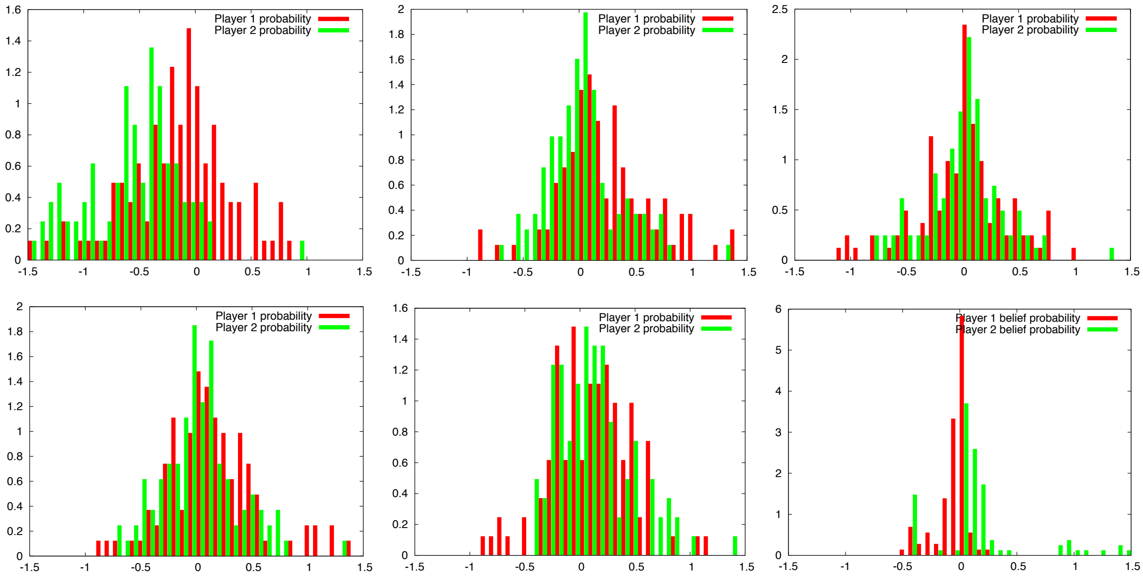

FIG. 5: Distribution of the error parameter, $\log(r_1)$, for the Selten and Chmura (2008) data when compared to the predictions of various equilibrium concepts. The top row of panels show, from left to right, the Nash equilibrium, the QRE and the Action Sampling equilibrium. The bottom row of panels show, from left to right, the Payoff Sampling equilibrium, the Impulse Balance equilibrium and, for comparison, the RBB equilibrium described in this paper. We note that in each bin of the histogram the bars for player 1 and player 2 have been deliberately offset from one another for clarity, but refer to the same range of values in the bin.

Brunner et al. (2010) we compute the QRE with $\lambda = 1.05$, the action sampling equilibrium with $n = 12$ and the payoff-sampling equilibrium with $n = 6$. In Table I we show p-values comparing these mean squared errors between the models using the Wilcoxon signed-rank matched-pairs test. Where a significant difference was found, we also did a one-sided test on the same data to see which model predicted significantly smaller errors. The results of these one-sided tests are also shown in the table. For these tests we use only data from Games 1-6, 7, 9, 11 and 12 due to the issue in computing the RBB equilibrium solution in Games 8 and 10. These p-values strikingly reinforce the conclusion we drew from the error distributions — the RBB equilibrium solution is significantly better at explaining the data than the other equilibrium concepts. We used the Wilcoxon test to allow direct comparison to Selten and Chmura (2008) and Brunner et al. (2010). Brunner et al. (2010) point out correctly that the Wilcoxon test is a parametric test, and the underlying distributional assumptions may not apply to the Selten and Chmura data. Brunner et al. (2010) also presented results based on the non-parametric Kolmogorov-Smirnov and robust rank-order tests. Selten, Chmura and Goerg (2011) noted that the Kolmogorov-Smirnov test is also not entirely appropriate here, as it is not a matched-pairs test. They suggested the Fisher-Pitman test which is a non-parametric test for matched pair comparisons. Although the Kolmogorov-Smirnov statistical test is non-parametric, it does tell us whether the distributions of mean-squared errors are significantly different between the equilibrium concepts. We performed the Kolmogorov-Smirnov test and the results were consistent with the conclusions from the Wilcoxon test. We also performed the Fisher-Pitman test and those results are shown in Table II. These results are also consistent with the conclusions from the Wilcoxon test and support the conclusion that the RBB equilibrium solution is significantly better at explaining the data than the other equilibrium concepts.

In summary, we have demonstrated in this section how the hyperprior concept provides a natural way to characterize departures from the equilibrium solution observed in experiments. Our framework attempts to split up the deviation into pieces that are player dependent (the X_i's) and pieces that are game dependent. We have found that the player-dependent pieces are well fit by a log-normal distribution and that this kind of model can explain experimental data very closely. This type of analysis can be readily adapted to other equilibrium concepts and games. For data in Selten and Chmura (2008), the RBB model appears to explain the data significantly better than other equilibrium concepts when the comparison is done in terms of belief probabilities, which is natural in the RBB framework.

IV. CONCLUSION

Scholars in statistics and economics have long been interested in the empirical testing of game-theoretic anomalies. A sizeable experimental literature shows that there is a dissonance between the theory and empirics of games in a strategic setting where defection is predicted by theory to be the optimal outcome over cooperation. Players in experimental settings cooperate far more than the theory would predict.

In this paper, we provide a reason for this unreason and explain this anomaly by providing a new method termed the 'returns-based beliefs' approach in which players form subjective beliefs and then act based upon the expected returns, given these beliefs, of a particular strategy, in proportion to the total expected returns of all strategies. Our approach treats

	Payoff sampling	QRE	Impulse balance	RBB	Nash
Action	n.s.	n.s.	n.s.	0.1% (g: 0.1%)	0.1% (l: 0.1%)
Sampling	n.s.	10% (l: 5%)	5% (l: 5%)	0.1% (g: 0.1%)	0.1% (l: 0.1%)
($n = 12$)	n.s.	n.s.	n.s.	5% (g: 2%)	1% (l: 0.5%)
Payoff		n.s.	n.s.	0.1% (g: 0.1%)	0.1% (l: 0.1%)
Sampling		0.1% (l: 0.1%)	0.1% (l: 0.1%)	0.1% (g: 0.1%)	0.1% (l: 0.1%)
($n = 6$)		0.5% (g: 0.5%)	2% (g: 1%)	5% (g: 5%)	2% (l: 1%)
QRE			10% (l: 5%)	0.1% (g: 0.1%)	0.1% (l: 0.1%)
($\lambda = 1.05$)			0.2% (l: 0.1%)	0.1% (g: 0.1%)	0.1% (l: 0.1%)
			10% (g: 5%)	10% (g: 5%)	2% (l: 1%)
Impulse				0.1% (g: 0.1%)	0.1% (l: 0.1%)
Balance				0.1% (g: 0.1%)	0.1% (l: 0.1%)
				n.s. (g: 10%)	1% (l: 0.5%)
					0.1% (l: 0.1%)
RBB					0.1% (l: 0.1%)
					0.2% (l: 0.1%)

TABLE I: P-values in favour of row concepts, using two-tailed matched-pairs Wilcoxon signed rank test (rounded to the next higher level among 0.1%, 0.2%, 0.5%, 1%, 2%, 5% and 10%). Where results are significant for the two-tailed test, we quote the significance of the one-sided test in brackets, with "l" indicating the mean of the error distribution for the row value is significantly lower than that of the the column value, and "g" indicating it is significantly greater. In each entry of the table, the first row uses data from Games 1-6, 7, 9, 11 and 12; the middle row uses data from Games 1-6 only; the last row uses data from Games 7, 9, 11 and 12 only.

	Payoff sampling	QRE	Impulse balance	RBB	Nash
Action	n.s.	n.s.	n.s.	0.1% (g: 0.1%)	0.1% (l: 0.1%)
Sampling	n.s.	n.s.	10% (l: 5%)	0.1% (g: 0.1%)	0.1% (l: 0.1%)
($n = 12$)	n.s. (l: 10%)	n.s.	n.s. (g: 10%)	1% (g: 0.5%)	1% (l: 0.5%)
Payoff		n.s.	n.s.	0.1% (g: 0.1%)	0.1% (l: 0.1%)
Sampling		0.1% (l: 0.1%)	0.5% (l: 0.5%)	0.1% (g: 0.1%)	0.1% (l: 0.1%)
($n = 6$)		0.5% (g: 0.5%)	0.5% (g: 0.2%)	1% (g: 0.5%)	2% (l: 1%)
QRE			n.s.	0.1% (g: 0.1%)	0.1% (l: 0.1%)
($\lambda = 1.05$)			n.s.	0.1% (g: 0.1%)	0.1% (l: 0.1%)
			2% (g: 1%)	2% (g: 1%)	2% (l: 1%)
Impulse				0.1% (g: 0.1%)	0.1% (l: 0.1%)
Balance				0.1% (g: 0.1%)	0.1% (l: 0.1%)
				5% (g: 5%)	2% (l: 1%)
					0.1% (l: 0.1%)
RBB					0.1% (l: 0.1%)
					0.1% (l: 0.1%)

TABLE II: P-values in favour of row concepts, using the Fisher-Pitman test (rounded to the next higher level among 0.1%, 0.2%, 0.5%, 1%, 2%, 5% and 10%). Where results are significant for the two-tailed test, we quote the significance of the one-sided test in brackets, with "l" indicating the mean of the error distribution for the row value is significantly lower than that of the the column value, and "g" indicating it is significantly greater. In each entry of the table, the first row uses data from Games 1-6, 7, 9, 11 and 12; the middle row uses data from Games 1-6 only; the last row uses data from Games 7, 9, 11 and 12 only.

beliefs rather than strategies as the primary concept. In so doing, we assume that the players' beliefs and actions are in equilibrium. Our approach combines a decision analytic solution concept where individuals form subjective probabilities over the actions of the individual's opponent and then choose a mixed strategy profile over the actions, using the probabilistic choice model developed axiomatically by Luce (1959), that is based on the relative returns of each strategy given these subjective probabilities. Our approach is different to the Nash equilibrium as we (1) replace best replying by each player with the Luce (1959) probabilistic choice model; (2) introduce probabilistic beliefs; and (3) treat the actions/beliefs as uncertain rather than random. In doing so, as discussed in the Introduction section, we are able to distinguish the RBB equilibrium with other non-Nash equilibrium concepts such as the QRE, BRNE and RBE. We hypothesize that when agents form subjective beliefs about the strategies of the other players then there might be opportunities for profitable deviation from the Nash equilibrium strategies when the other player is expected to deviate from playing the Nash strategy. We incorporate the concept of hyperpriors to account for the ambiguity of the cooperative stance of the players.

We test our model on various classes of games: for example, we show how our approach provides a closer description of empirical observations of the Prisoner's Dilemma. Our approach explains two anomalies between the theoretical predictions of one-shot Prisoner's Dilemma game and the empirical evidence: first, the inverse relationship between cooperation and the temptation to defect; and second, the positive relationship between cooperation and the gains from cooperation.

In addition, we test the closeness of fit of our model using data from Selten and Chmura (2008) for completely mixed 2×2 games. In particular, we show that the returns-based belief model is able to explain the empirical data from completely mixed 2×2 games once loss aversion is included, with small deviations about the equilibrium solution. We show that the RBB equilibrium is generally better at explaining the data than the other stationary concepts discussed in Selten and Chmura (2008), showing a much narrower distribution in the deviation parameter between the actual and the predicted values and a significantly smaller mean-square error. We would like to emphasize that, in contrast to the other equilibrium concepts discussed in the paper, the RBB equilibrium is not based on fitting the data in order to obtain the best fit of a free parameter. Assessing the RBB equilibrium involves comparing the beliefs that need to be held by the player against the predicted belief of the model.

It is important to acknowledge that there are some limitations of the returns-based beliefs equilibrium. First, the model does not allow for negative returns in its basic set up. However, we provide one possibility to overcome this shortcoming through the zeroing algorithm discussed earlier. Second, the Luce rule suffers from the well known regularity of the duplicates problem outlined by Debreu, whereby the addition of duplicates to the choice set might alter the probability of choosing them in such a way that is inconsistent (Gul, Natenzon and Pesendorfer 2010). However, Gul, Natenzon and Pesendorfer (2010) propose a modification of the Luce rule whereby the attributes weighted by their value overcomes the duplicates problem. Third, the Luce rule might violate first order stochastic dominance (see Blavatskyy 2011). Blavatskyy (2011) proposes a model of probabilistic choice that satisfies first order stochastic dominance. Identifying these caveats provides avenues for further refinement of the returns-based beliefs equilibrium concept.

Acknowledging these limitations, we argue that returns-based beliefs are useful in game theory because they allow us to reconcile game theoretic predictions and empirical experimental results. The sensitivity of the equilibrium to changes in payoff is an important aspect of our model that accords more with reality than the conventional Nash equilibrium. Moreover, even if returns-based beliefs do provide predictions in some situations, there may be other avenues for modeling non-Nash equilibrium behavior, which when combined with our model may provide additional insights. Therefore, we hope that this study provides a helpful prelude for better understanding the impulses which ultimately govern human behavior.

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