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ABSTRACT

Lotteries vs. All-Pay Auctions in Fair and Biased Contests*

The form of contests for a single fixed prize can be determined by a designer who maximizes the contestants' efforts. This paper establishes that, under common knowledge of the two asymmetric contestants' prize valuations, a fair Tullock-type endogenously determined lottery is always superior to an all-pay-auction; it yields larger expected efforts (revenues) for the contest designer. If the contest can be unfair (structural discrimination is allowed), then the designer's payoff under the optimal lottery is equal to his expected payoff under the optimal all-pay auction.

JEL Classification: D70, D71, D72

Keywords: contest design, efforts (revenue) maximization, discrimination, endogenous lottery, all-pay auction

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1. Introduction

Many economic and political decisions are the outcome of all-pay strategic contests for a given prize. In such contests the single winner depends on the efforts invested by the contestants. Applications of these contests include promotional competitions, litigation, internal labor market tournaments, rent-seeking, R&D races, political and public policy competitions and sports, Epstein and Nitzan (2007), Konrad (2009), Congleton et al. (2008). Endogenous determination of such contests may involve all of their relevant institutional characteristics. These are typically determined by contest designers; economic and political entrepreneurs who wish to maximize the efforts made by the contestants. Most of the literature on optimal contest design has focused on the choice of the contest prize, the set of contestants, the structure of multi-stage contests, caps on political lobbying, and the contest success function, CSF, the function that relates the contestants' efforts to their winning probabilities. In the current study we focus on the design of logit CSF's that include the two most widely studied types of mechanisms: Tullock's type lotteries and all-pay-auctions. We show that in asymmetric contests with different prize valuations, the endogenously determined lottery always yields certain efforts that are larger than or equal to the expected efforts in an APA.

We first study fair non-discriminating logit CSF's, as in Alcalde and Dahm (2010) and Nti (2004), allowing control of the exponent determining the particular form of the logit CSF. In this context, Fang (2002) has shown that a fair simple lottery can be superior to an all-pay-auction and induce larger efforts, if the gap between the contestants' prize valuations is sufficiently large.[1] Our main result considerably extends his finding. It establishes that *the optimal* fair (non-discriminating) lottery is *always* superior to the all-pay-auction (APA). That is, it yields larger efforts regardless of the gap between the contestants' stakes. We then extend the setting by allowing discrimination, that is, control of another parameter that determines the preferential treatment received by one of the contestants, as in Lien (1986), (1990), Clark and Riis (2000) and, more recently, Franke (2012), Epstein et al. (2011). Such discrimination is commonly observed in real political-economic contest environments, and, particularly, in the public sector (For a detailed discussion of the empirical relevance of discrimination and its control by contest designers see Epstein

[1] See also Epstein and Gang (2009).

et al. (2011) and Franke et al. (2011)). By our second result, when discrimination is allowed, the designer's payoff under the optimal discriminating lottery is equal to his expected payoff under the optimal discriminating APA.

Given our assumption of common knowledge of the contestants' prize valuations, the contest designer could resort to a mechanism that completely extracts the higher prize valuation. Focusing on contests with logit CSF's actually implies that we rule out such mechanisms. In other words, the contestants in our setting are in fact assumed to be protected from complete extraction of their surplus because they are ensured that in the contest equilibrium their participation is minimally effective; in equilibrium every contestant makes a positive effort with positive probability and has a positive probability of winning the contested prize.[2] The focus on logit lotteries conforms therefore to a legal constraint that contests induce participation and that the designer cannot induce efforts that exceed, as we shall see, the average value of the contested prize.

It should be noted that the logit CSFs can be justified either axiomatically or on the grounds of common use in practice. In any event, they are the most widely studied functions in the contest literature. Despite their popularity, it must be admitted that it remains an open question whether some other CSFs, while still ensuring minimally effective participation of the contestants, can yield better results for the contest designer. Nevertheless, and more importantly, since we prove the superiority of Tullock-type lotteries over the APA in fair contests, and since we prove the equivalence of the optimal lottery and the optimal APA in unfair-discriminating contests, which implies that a risk averse designer would prefer the optimal lottery, our assumption regarding the restriction imposed on the designer (the selection of a CSF that belongs to the particular family of a logit lotteries), is justified because it is sufficient to raise doubt regarding the apparent belief regarding the superiority of APA as a means of generating revenue for the contest designer.

The remainder of the paper is laid out as follows. In Section 2, we present the model, the optimal contest design approach that allows discrimination and the two types of contest success functions, APAs and Tullock's logit lotteries. Section 3 contains the main result which establishes the inferiority of the APA relative to the

[2] The existence of effective incentives of participation precludes the direct abolition of competition by exclusion of contestants or the indirect abolition of competition by application of a Tullock CSF that for some contestants is always unresponsive to their effort, as in Nti (2004).

endogenous lottery, when the contestants have different prize valuations. The equivalence between the optimal lottery and APA under discrimination is established in Section 4. The novelty of our contribution relative to the literature is clarified in Section 6. Concluding remarks are included in Section 6. All the proofs are relegated to an Appendix.

2. Optimal contest design

a. The setting

In the basic one-stage contest setting, there are two risk-neutral contestants, the high and low benefit contestants, 1 and 2. The prize valuations of the contestants are denoted by n_i, $n_1 > n_2$ or $k = \frac{n_1}{n_2} > 1$. We assume that the designer has full knowledge of the contestants' prize valuations. Given these valuations and the CSF, the function that specifies the contestants' winning probability given their efforts, $\Pr_i(x_1, x_2)$, the expected net payoff of contestant *i* is:

$$E(u_i) = \Pr_i(x_1, x_2)n_i - x_i, \ (i{=}1{,}2) \tag{1}$$

where x_1 and x_2 denote the contestants' efforts. In the extended optimal contest design setting, the objective function of a third player, the designer of the contest, is:

$$G = E(x_1 + x_2) \tag{2}$$

The contest designer is assumed to maximize his objective function (2) by setting the CSF, anticipating the Nash equilibrium efforts of the contestants that are obtained in the standard contest where the payoff functions of the contestants are given by (1) and the CSF is set by the designer. As already mentioned in the introduction, we focus on the widely studied CSFs that include APAs and Tullock's lotteries.[3]

b. All-Pay Auctions

Under an APA, the certain winner is the contestant who makes the largest effective effort, where a unit of effort by one contestant is not necessarily equally effective as a unit of effort of his rival, as first suggested in the context of a bribery game by Lien

[3] In Epstein et al. (2011), when the weight assigned to the expected welfare of the contestants is sufficiently high, in equilibrium there is no contest (and so no efforts are made) and the winner is the contestant with the higher valuation. In contrast, in our model, such equilibrium cannot emerge since the weight assigned to the expected welfare of the contestants is zero. Therefore, any equilibrium is an interior one: there is real competition and "meaningful" winning (each contestant makes an effort with a positive winning probability).

(1986), (1990) and later on by Clark and Riis (2000). That is, the CSF for $\delta > 0$ is an APA given by:

$$p_1(x_1, x_2) = \begin{cases} 1 & \text{if} \quad x_1 > \delta x_2 \\ 0.5 & \text{if} \quad x_1 = \delta x_2 \\ 0 & \text{if} \quad x_1 < \delta x_2 \end{cases} \tag{3}$$

and for $\delta = 0$, $p_1(x_1, x_2) = 1$, where the discrimination variable $\delta \geq 0$ is selected by the contest designer. By (3), a reduction in δ increases the bias in favor of the more motivated contestant 1. Furthermore, $0 \leq \delta < 1$ implies a bias in favor of contestant 1. When $\delta = 1$ the contest is fair, there is no bias. When $\delta > 1$ the bias is in favor of contestant 2.

c. The logit lotteries

Under a lottery, every contestant has some positive winning probability. Sufficiently large investment of effort can secure a high probability of winning, but not certain winning. Our optimality and neutrality results are confined to the well studied logit Tullock-type lotteries. For $\delta > 0$, these lotteries are given by:[4]

$$p_1(x_1, x_2) = \frac{x_1^\alpha}{x_1^\alpha + (\delta x_2)^\alpha} \tag{4}$$

$0 < \alpha < \infty$ and for $\delta = 0$, $p_1(x_1, x_2) = 1$, where α and δ are selected by the contest designer. The interpretation of δ is as in sub-section (b).

3. Fair contests

As in Alcalde and Dahm (2010) and Nti (2004), in the standard environments the designer cannot discriminate between the contestants, $\delta = 1$.

a. The unbiased APA

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[4] As is well known, see Konrad (2009) and references therein, when $0 < \alpha \leq 2$ the contest has a pure-strategy equilibrium. Since $p_1(x_1, x_2) = \frac{x_1^\alpha}{x_1^\alpha + (\delta x_2)^\alpha} = \frac{1}{1 + (\delta x_2 / x_1)^\alpha}$, one can easily see that for $\alpha = \infty$ the logit lottery takes the form (3). The form in (4) is slightly different from the form of the logit lottery in Epstein et al. (2011) because in the present study, as will become clear in the sequel, it is important to make a meaningful comparison between the degree of discrimination under the two types of CSFs. Such comparison requires that discrimination is defined in a similar way in the two cases.

In equilibrium of the unbiased APA, Hillman and Riley (1989), Konrad (2009), the expected efforts are $x_1^* = 0.5n_2$ and $x_2^* = \frac{n_2^2}{2n_1}$. In turn, the expected net payoff of the player with the lower prize valuations is zero, namely, when $k > 1$, only the player with the higher prize valuations enjoys some surplus. The expected net payoff of the player with the higher prize valuations is equal to $E(u_1^*) = (n_1 - n_2)$. In equilibrium, the value of the designer's objective function (the expected aggregate efforts of the contestants in the mixed-strategy equilibrium) is equal to $G_A = \frac{n_2(n_1 + n_2)}{2n_1}$.

b. The optimal unbiased lottery

With asymmetric prize valuations and $\alpha = 1$, it is known that in equilibrium $G_L = \frac{n_1 n_2}{n_1 + n_2}$ is larger than G_A, provided that the gap between the contestants' stakes is sufficiently large, that is, $k > 1 + \sqrt{2}$, see Fang (2002). Our first result considerably strengthens this finding by establishing that in a fair contest, if $k > 1$, then a designer who can select the exponent α, *always* prefers a lottery because it yields larger efforts relative to the APA, even when Fang's sufficient condition *is not* satisfied.

Proposition 1:

If discrimination is not feasible, $k > 1$ and the designer can select the exponent α, then there exists a value in the range $0 < \alpha < 2$ that yields certain contestants' efforts that are larger than the expected efforts obtained under the APA.

(For a proof, see Appendix A).

This result implies that, in a fair contest, if $k > 1$, then a Tullock-type lottery is always preferred to the APA by a risk neutral or a risk averse designer. Alcalde and Dahm (2010) have shown that for any $\alpha \geq 2$ there exists an equilibrium in mixed strategies that is equivalent to the equilibrium of the APA. However, so far a characterization of the complete set of mixed-strategy equilibria is not available. Since we show that there exists α, $0 < \alpha < 2$, that yields efforts that are larger than those obtained under the APA, it is clear that this conclusion remains valid when the parameter α has to satisfy the requirement $0<\alpha<\infty$.

Let us explain the economic intuition behind the result, namely, why $k>1$ gives rise to a pure-strategy equilibrium corresponding to $0<\alpha(k)<2$ which is preferred to the mixed-strategy equilibrium obtained under the APA (the logit CSF where $\alpha=\infty$).

For $k>1$, a designer setting an optimal α for a pure-strategy equilibrium must choose an exponent α which is smaller than 2. Furthermore, in a pure-strategy equilibrium the designer may choose an exponent that is smaller than that α which causes contestant 2's utility to be equal to zero. In other words, in equilibrium contestant 2's utility can be positive. This has been shown by Nti (2004) for a sufficiently large value of *k* (see Section 4 in his paper). However, for lower values of *k*, but still $k>1$, α reduces contestant 2's utility to zero.[5] In any case, as explained above, for $k>1$ the optimal α for a pure-strategy equilibrium is smaller than 2, $0<\alpha(k)<2$.

Under the APA where $\alpha=\infty$, the contest equilibrium is in mixed strategies, the total expected efforts are equal to $G_L=\dfrac{n_2(n_1+n_2)}{2n_1}$ and the surplus of contestant 2 is completely eliminated. It remains to explain why the designer prefers the optimal pure-strategy equilibrium to the mixed-strategy equilibrium. Notice that in the move from equilibrium in pure strategies to equilibrium in mixed strategies α is increased, however, $x_1^*/x_2^*=k$.[6] The increase in α increases the winning probability of contestant 1 and reduces the winning probability of contestant 2. However, contestant 1 is induced to reduce his efforts in the mixed-strategy equilibrium corresponding to the larger α because this further increases his expected utility. Such reduction is

[5] The critical *k* is $k=3.509$.

[6] By substituting $\delta=1$ in equation (3.B) in Appendix B, we get that the contestants' efforts in a pure-strategy equilibrium are equal to $x_{1p}^*=\dfrac{\alpha n_1 k^\alpha}{(k^\alpha+1)^2}$ and $x_{2p}^*=\dfrac{\alpha n_2 k^\alpha}{(k^\alpha+1)^2}$. Under the APA their expected efforts are $x_{1m}^*=0.5n_2$ and $x_{2m}^*=\dfrac{n_2^2}{2n_1}$ .Hence, $\dfrac{x_{1p}^*}{x_{2p}^*}=\dfrac{x_{1m}^*}{x_{2m}^*}=k$.

possible because contestant 2 whose utility is reduced or remains equal to zero in the mixed-strategy equilibrium is also induced to reduce his effort.[7]

Che and Gale (1997) point out that intuition suggests that a society with contest designers who are receptive to rent seeking (a large exponent α) would induce greater rent-seeking expenditures than other societies, all else equal. Our result establishes that this intuition is not valid when we move from the range $0 < \alpha \le 2$ to $\alpha = \infty$. In particular, when $\delta = 1$ and $k > 1$, the parameter α in the logit CSF on which we focus, that yields the largest efforts is not $\alpha = \infty$, since there is some α which is smaller than 2 that yields larger efforts.

4. Unconstrained contests

In the unconstrained environment that allows discrimination, anticipating the investments of the contestants, the designer optimally determines the parameters of the contest's success functions. When considering the APA the designer controls the degree of discrimination δ. When considering the logit lottery he determines both δ and the exponent α.

Most of the literature studying Tullock's lotteries and the APA disregarded deliberate discrimination between the contestants and control of the exponent α. However, in the context of an APA, Lien (1986), (1990) and Clark and Riis (2000) studied a bribery game in which a designer exercises discrimination in a multiplicative form. Michaels (1988) and Nti (2004) examined the effect of the exponent α on the contestants' efforts in Tullock' lotteries disregarding discrimination between the contestants and focusing on pure-strategy equilibria. Recently, Epstein et al. (2011) presented a complete analysis of the designer's effect on the contestants' efforts by control of the degree of discrimination both for the APA and the logit lotteries, assuming that in the lotteries' case the exponent α is given and restricted to the range $0 < \alpha \le 1$. They have thus re-shifted the emphasis in the study of optimal contest design to the control of discrimination. The current section combines the two approaches allowing the designer to control both the exponent α and the degree of discrimination between the contestants. In addition, it generalizes the analysis by allowing any value of the exponent α associated with a pure-strategy

[7] Recall that in the pure-strategy equilibrium the utility of contestant 2 can be positive if k is sufficiently large (see footnote 7).

or a mixed-strategy equilibrium, $0 < \alpha < \infty$, or to an APA that is associated with a mixed-strategy equilibria where $\alpha = \infty$.[8]

a. The optimal APA

Under the APA, the designer maximizes his objective function by determining the optimal value of δ. The optimal δ is equal to *k* (for a proof, see Epstein et al. (2011), Proposition 1, assuming that no weight is assigned to the expected welfare of the contestants). This optimal bias eliminates the advantage of contestant 1, creates actual equality between the competitors and completely eliminates their surplus. The corresponding value of the expected efforts of the contestants in the mixed-strategy equilibrium of the contest is $G_A = 0.5(n_1 + n_2)$.

b. The optimal lottery

Under the logit contest success function, the designer determines the optimal level of α, $0 < \alpha < \infty$, and $\delta, \delta \geq 0$. Let us partition the range of the parameter α into $0 < \alpha \leq 2$ and $2 < \alpha < \infty$. For $0 < \alpha \leq 2$, where a unique pure-strategy equilibrium exists, the optimal values of δ and α are equal to $\delta^* = k$ and $\alpha^* = 2$ (for a proof, see Appendix B. This proof has to take into account constraints that are not automatically satisfied as in Epstein et al. (2011) where $0 < \alpha \leq 1$. For further more specific clarification of this point see footnote 15).[9] The corresponding value of the certain efforts of the contestants in the pure-strategy equilibrium of the contest is $G_L = 0.5(n_1 + n_2) = G_A$.[10] We therefore obtain:

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[8] Note that under the logit CSF where $0 < \alpha \leq 2$ the exerted efforts are certain whereas under the APA the meaning of efforts is expected efforts.

[9] We suggest the following intuition regarding the equilibrium outcome and its sensitivity to α ($0 < \alpha \leq 2$). The designer tends to support the contestant with the lower prize valuation ($\delta^* = k$) to attain complete "balance" between the wining probabilities for any given α, $0 < \alpha \leq 2$. An increase in α induces the contestants to increase their effort at the same rate in order to increase their winning probability. Since at the same time the designer favorably discriminates the contestant with the lower prize valuation, the equilibrium winning probability is unchanged, $p_i = 0.5$. But by raising α the designer increases the aggregate efforts. The designer therefore prefers the highest possible α, $\alpha = 2$, that enables him to extract the maximal possible surplus from the contestants which reduces their net payoff to zero.

[10] Note that this result has the flavor of the neutrality result of Alcalde and Dahm (2010). However, in our setting of optimal contest design, the contestants' maximal efforts are larger than those obtained in the symmetric setting of Alcalde and Dahm (2010). This is due to the allowed control of both δ and α. Also note that the question whether the equilibrium aggregate effort in a contest with $\alpha > 2$ exceed that when $\alpha = 2$ or $\alpha = \infty$ is open, although we conjecture that it does not.

Proposition 2:

If the contest designer can select both the degree of discrimination δ, $\delta \geq 0$, and the exponent α, then the contestants efforts under the optimal lottery are equal to the expected efforts in the mixed-strategy equilibrium of the optimal APA.

The combined effect of optimal discrimination $\delta = k$ and optimal selection of the exponent α in the logit CSF

$$p_1(x_1, x_2) = \frac{x_1^\alpha}{x_1^\alpha + (\delta x_2)^\alpha} = \frac{1}{1 + (\delta x_2 / x_1)^\alpha} \tag{5}$$

that gives rise to a pure-strategy equilibrium when $0 < \alpha \leq 2$, results in equal (expected) efforts and elimination of the surplus of both of the contestants. This result implies that the optimal logit lottery (that need not satisfy the constraint $0 < \alpha \leq 2$) cannot be dominated by the APA.

Proposition 2 establishes that the intuition mentioned above that a larger exponent α would induce greater expenditures is, again, not valid. Under optimal discrimination, even the use of an optimal APA by designers who are maximally receptive to the contestants' efforts, would not induce greater efforts than those obtained by designers who are less receptive to the contestants' efforts, $\alpha^* = 2$, allowing random winning. Recall that in both cases bids are optimally discriminated, the optimal degree of discrimination being equal to *k*. Nevertheless, the valid part of the above intuition is that a larger exponent α indeed induces greater efforts under the logit lottery provided that the exponent α gives rise to a pure-strategy equilibrium.

5. Relationship to the literature

To clarify the novelty of our results, let us discuss the difference between the two settings we have focused on and the settings examined in Fang (2002), Epstein et al. (2011), Epstein and Nitzan (2006) and Franke et al. (2011).

Fang (2002) has shown that, under asymmetric prize valuations, a simple lottery yields larger efforts than the expected efforts under an APA provided that the gap between the contestants’ stakes is sufficiently large. Our first result, Proposition 1, considerably strengthens this result by establishing that in a fair contest ($\delta = 1$), if $k > 1$, then a designer who can select the exponent α, *always* prefers a logit lottery to an APA, even when Fang's sufficient condition *is not* satisfied. In fact, the exponent of the preferred lottery satisfies $0 < \alpha(k) < 2$, which means that the corresponding contest game has a unique pure-strategy equilibrium.

Epstein et al. (2011) have studied an extended setting where discrimination is allowed, the objective function of the contest designer is a weighted average of the contestants' expected welfare and their aggregate expenditures, however, the exponent α is given and restricted to $\alpha \le 1$. They have not dealt therefore with our first setting where the entire weight is put on efforts because we have not allowed discrimination. They have also not dealt with our second setting where the entire weight is put on efforts and discrimination is allowed because, first, we have allowed the contest designer to control α and, second, we have dealt with a less restricted α, $\alpha \le 2$. No wonder than that our two results differ from Proposition 5 in Epstein et al. (2011) that establishes the superiority of the optimal APA relative to any optimal lottery.

Epstein and Nitzan (2006) compared between the contestants' efforts under an APA and a lottery assuming that discrimination is not allowed where both α and k are given (clearly, the assumed α need not be the optimal exponent corresponding to the given parameter *k*). The first settings in the current paper is fundamentally different because the exponent α is assumed to be optimally determined by the contest designer, given the parameter *k*. For this reason our Proposition 1 and the result reported in section 4.2.2 in Epstein and Nitzan (2006) are different.[11]

Franke et al. (2011) have recently compared the performance of the optimal APA and the optimal simple lottery in an extended n-player contest with discrimination. They have been able to prove the superiority of the APA assuming a simple lottery, viz., $\alpha = 1$. They have not allowed, however, control of the exponent α as in our second setting where $\alpha \le 2$. Hence their main result differs from our Proposition 2 that establishes the equivalence between the performance of the optimal APA and the optimal lottery.

6. Conclusion

a. Extraction of the contestants' surplus and the designer's payoff

In the unconstrained environment that allows discrimination, the designer always captures all the surplus of the two contestants. In the constrained fair environment, this is not the case although under the APA the designer always captures the surplus

[11] In fact, the latter result has to be amended because it disregards the non-negativity constraints on the contestants' utilities as in our derivation of Proposition 1.

of contestant 2. The difference between the two competitive environments is the extent of the contestants' incentives to make efforts. This hinges on the possibility of discrimination between the contestants. The possibility of discrimination increases the intensity of competition enabling the designer to increase his payoff.

b. The shadow price of the competitiveness constraint

The competitiveness constraint means that we deal with interior contest equilibria such that the ability of the contest designer to induce efforts is limited to the average value of the contested prize. Without this constraint, the designer could yield a total effort of n_1 by excluding the rivals of the contestant with the maximal valuation of the contested prize and then exploiting his political power or bargaining advantage to extract (almost) all his surplus, as in Nti (2004). We should point out that several different mechanisms can be used to implement the optimal design. For example, we could use an auction mechanism where the prize is awarded to the highest bidder but the contest designer has a reserve price equal to n_1 (see Glazer, 1993) or, alternatively, use a first-price APA with a reservation price of n_1 (see Hillman and Riley, 1989). The CSFs on which we focus do conform to the competitive environment and the maximal efforts are equal to $0.5(n_1 + n_2)$. This result is consistent with the constraint on the ability of the designer to induce efforts being binding. The shadow price of the competitiveness constraint is therefore equal to $n_1 - 0.5(n_1 + n_2) = 0.5(n_1 - n_2)$. This is then the price of competition for the designer.

c. The shadow price of the equalitarian constraint

Equalitarianism results in reduced payoff for the designer relative to the unconstrained environment when $k > 1$. In this case the shadow price of this constraint is positive, the exact value depending on the gap between the contestants' prize valuations that gives a Tullock's lottery a clear-cut advantage over the APA, as implied by Proposition 1. This may certainly support an ideology protecting the equalitarian 'welfare state'. Simply, in our setting, such equalitarianism limits the ability of the political-economic entrepreneur to extract resources from the contestants.

d. Generalization to n-player contests

A potential interesting extension of our study is the analysis of the multiple-player case. Results for the all-pay auction should be robust with respect to the number of players because only two players will actively participate in equilibrium. This is not the case for lotteries. However increasing the number of contestants intensifies competition so our results might remain valid for more than two contestants both in fair and unfair contests. The case of optimal discrimination by control of the bias scheme has been recently analyzed in Franke et al. (2011) assuming the simple lottery case where $\alpha = 1$; the contestants' winning probabilities are equal to their relative exerted efforts. Even the analysis of this relatively simple case is rather complex. Its extension to our more general setting of Tullock lotteries, where $0 < \alpha \leq 2$, seems a worthwhile yet an especially demanding challenge.

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Appendix A: Proof of Proposition 1

When discrimination is not allowed, $\delta = 1$, and the exponent α is chosen optimally taking the form $\alpha(k)$, by Nti (2004), in the range $0 < \alpha \le 2$ the pure-strategy equilibrium results in[12]

$$G_L^{ps} = \frac{\alpha k^{\alpha}\left(n_1 + n_2\right)}{(k^{\alpha} + 1)^2}$$

This result is obtained assuming that the second order conditions for maximization are satisfied and that the contestants' utilities are not negative. In this case the contestants' efforts, winning probabilities and utilities are equal to those specified in Appendix B, substituting $\delta = 1$ in 3.B-5.B. Since the second order conditions and the conditions ensuring the non-negativity of the contestants' utilities are given by 6.B-

[12] Notice that Nti defines *k* as the inverse of our *k*, that is, in his case $k = n_2/n_1$. Consequently, his result is presented with the appropriate modification.

7.B, the designer's problem is given by 8.B (substituting $\delta = 1$). Since $\delta = 1$ the third constraints in 8.B is satisfied. The first constraint is satisfied (the utility of contestant 1 is not negative) if the second constraint is satisfied (the utility of contestant 2 is not negative). Therefore, when $\delta = 1$, the designer maximizes G_L^{ps} only subject to the second constraint, $(\alpha - 1)k^{\alpha} \leq 1$.

Under the APA, where $\alpha = \infty$, the mixed-strategy equilibrium results in

$$G_A = \frac{n_2(n_1 + n_2)}{2n_1} = \frac{n_1 + n_2}{2k}$$

Denote by $\bar{\alpha}$, which is a function of the parameter *k*, the solution of

$$(\alpha - 1)k^{\alpha} = 1 \tag{1.A}$$

Under such equality the expected utility of contestant 2 is equal to zero. Equality (1.A) implies that $\bar{\alpha} > 1$, $\bar{\alpha} k^{\bar{\alpha}} = k^{\bar{\alpha}} + 1$, $\frac{\partial \bar{\alpha}}{\partial k} < 0$, $\bar{\alpha}(k = 1) = 2$ and ($k \to \infty$) $\Rightarrow$($\bar{\alpha} \to 1$), Hence, for every $k > 1$, $1 < \bar{\alpha} < 2$.

Nti (2004, Theorem 3) has shown that for $k \leq 3.509$, the solution (in implicit form) of equation (1.A), $\bar{\alpha}$, is the optimal solution that maximizes G_L^{ps}. Otherwise the solution differs from $\bar{\alpha}$. This means that

$$Max\, G_L^{ps} \geq \frac{\bar{\alpha} k^{\bar{\alpha}}(n_1 + n_2)}{(k^{\bar{\alpha}} + 1)^2} = \frac{\bar{\alpha} k^{\bar{\alpha}}(n_1 + n_2)}{\left(\bar{\alpha} k^{\bar{\alpha}}\right)^2} = \frac{n_1 + n_2}{\bar{\alpha} k^{\bar{\alpha}}}$$

Therefore, when $\frac{n_1 + n_2}{\bar{\alpha} k^{\bar{\alpha}}} > G_A$ we get that $Max\, G_L^{ps} > G_A$. That is,

$$\frac{n_1 + n_2}{\bar{\alpha} k^{\bar{\alpha}}} > \frac{n_1 + n_2}{2k}$$

or

$$2k^{1-\bar{\alpha}} > \bar{\alpha}$$

By (1.A), $k = \left(\bar{\alpha} - 1\right)^{-\frac{1}{\bar{\alpha}}}$. Substituting *k* in the latter inequality we get:

$2\left(\bar{\alpha} - 1\right)^{\frac{\bar{\alpha}-1}{\bar{\alpha}}} > \bar{\alpha}$ or $\left(\bar{\alpha} - 1\right)^{\frac{\bar{\alpha}-1}{\bar{\alpha}}} > 0.5\bar{\alpha}$ or :

$$\left(\bar{\alpha} - 1\right)^{\bar{\alpha}-1} - \left(0.5\bar{\alpha}\right)^{\bar{\alpha}} > 0 \tag{2.A}$$

Notice that if inequality (2.A) holds for $1 < \bar{\alpha} < 2$, then it is satisfied for every $k > 1$.[13] The reason is that an increase in k reduces $\bar{\alpha}$ and for $k = 1$, $\bar{\alpha} = 2$ whereas for $k \to \infty$, $\bar{\alpha}$ converges to 1.

Let $f(\bar{\alpha}) = (\bar{\alpha} - 1)^{\bar{\alpha}-1}$ and $g(\bar{\alpha}) = (0.5\bar{\alpha})^{\bar{\alpha}}$. To prove inequality (2.A), we have to show that for $1 < \bar{\alpha} < 2$, $f(\bar{\alpha}) > g(\bar{\alpha})$.[14]

Consider first the two functions f and g at $1 < \bar{\alpha} \le 2$. It can be verified that $\lim_{\bar{\alpha} \to 1} f(\bar{\alpha}) = 1$, $f(2) = 1$ and $\frac{df(\bar{\alpha})}{d\bar{\alpha}} = (\bar{\alpha} - 1)^{\bar{\alpha}-1}\left[1 + \ln(\bar{\alpha} - 1)\right]$. Therefore, for $1 < \bar{\alpha} \le 2$, f is minimized at $\bar{\alpha} = 1 + e^{-1}$, where $\min f(\bar{\alpha}) = (e^{-1})^{e^{-1}} = 0.6922$. In contrast, for $1 < \bar{\alpha} \le 2$, we get that $\frac{dg(\bar{\alpha})}{d\bar{\alpha}} = (0.5\bar{\alpha})^{\bar{\alpha}}\left[1 + \ln(0.5\bar{\alpha})\right] > 0$. That is, the function g is increasing in the domain $1 < \bar{\alpha} \le 2$.

Partitioning this domain, $1 < \bar{\alpha} \le 2$, into two subsets: $1 < \bar{\alpha} \le 1 + e^{-1}$ and $1 + e^{-1} < \bar{\alpha} \le 2$, let us show that in each of these sub-domains inequality (2.A) is satisfied.

Consider first the sub-domain $1 < \bar{\alpha} \le 1 + e^{-1}$. Since at $\bar{\alpha} = 1 + e^{-1}$, $g(1 + e^{-1}) = 0.5947 < 0.6922 = \min f(\bar{\alpha})$, and since, as noted above, g is increasing in the domain $1 < \bar{\alpha} \le 2$, we get that for $1 < \bar{\alpha} \le 1 + e^{-1}$, $f(\bar{\alpha}) > g(\bar{\alpha})$.

To complete the proof, let us show that the latter inequality is also satisfied for $1 + e^{-1} < \bar{\alpha} < 2$. In this subset of the domain of f and g, $0.5\bar{\alpha} > \bar{\alpha} - 1 > e^{-1}$ or $\ln(0.5\bar{\alpha}) > \ln(\bar{\alpha} - 1) > \ln e^{-1}$ or $\ln(0.5\bar{\alpha}) > \ln(\bar{\alpha} - 1) > -1$ or $1 + \ln(0.5\bar{\alpha}) > 1 + \ln(\bar{\alpha} - 1) > 0$ or $\frac{1 + \ln(0.5\bar{\alpha})}{1 + \ln(\bar{\alpha} - 1)} > 1$. Note that when $\bar{\alpha} = 2$, $\frac{1 + \ln(0.5\bar{\alpha})}{1 + \ln(\bar{\alpha} - 1)} = 1$. To sum up, when $1 + e^{-1} < \bar{\alpha} < 2$, we get that $\frac{1 + \ln(0.5\bar{\alpha})}{1 + \ln(\bar{\alpha} - 1)} > 1$ and when $\bar{\alpha} = 2$, $\frac{1 + \ln(0.5\bar{\alpha})}{1 + \ln(\bar{\alpha} - 1)} = 1$.

[13] When $k = 1$, $\bar{\alpha} = 2$ and we get that $Max\, G_L^{ps} = G_A = 0.5(n_1 + n_2) = n_1$.

[14] We are indebted to Arkadi Koziashvili for his help in establishing this part of the proof.

Let $h(\bar{\alpha}) = f(\bar{\alpha}) - g(\bar{\alpha})$. This function is continuous and differentiable in the interval $1+e^{-1} \le \bar{\alpha} \le 2$ and $\frac{dh(\bar{\alpha})}{d\bar{\alpha}} = f(\bar{\alpha})[1+\ln(\bar{\alpha}-1)] - g(\bar{\alpha})[1+\ln(0.5\bar{\alpha})]$. When the function $h(\bar{\alpha})$ has an extremum, $\frac{dh(\bar{\alpha})}{d\bar{\alpha}} = 0$ or $f(\bar{\alpha})[1+\ln(\bar{\alpha}-1)] - g(\bar{\alpha})[1+\ln(0.5\bar{\alpha})] = 0$ or:

$$\frac{f(\bar{\alpha})}{g(\bar{\alpha})} = \frac{1+\ln(0.5\bar{\alpha})}{1+\ln(\bar{\alpha}-1)} \tag{3.A}$$

We therefore get that:

1. The latter equality is satisfied for $\bar{\alpha} = 2$ which means that this value is an extremum value. In addition, for this extremum value, $f(\bar{\alpha}) = g(\bar{\alpha})$ or $h(\bar{\alpha}) = 0$.
2. As noted above, when $1+e^{-1} < \bar{\alpha} < 2$, we get that $\frac{1+\ln(0.5\bar{\alpha})}{1+\ln(\bar{\alpha}-1)} > 1$ and, in addition, by (3.A), $\frac{f(\bar{\alpha})}{g(\bar{\alpha})} = \frac{1+\ln(0.5\bar{\alpha})}{1+\ln(\bar{\alpha}-1)}$. Therefore, every extremum value in this sub-domain, $1+e^{-1} < \bar{\alpha} < 2$, satisfies the inequality $\frac{f(\bar{\alpha})}{g(\bar{\alpha})} = \frac{1+\ln(0.5\bar{\alpha})}{1+\ln(\bar{\alpha}-1)} > 1$ or $\frac{f(\bar{\alpha})}{g(\bar{\alpha})} > 1$. Hence, every extremum value in this sub-domain satisfies $f(\bar{\alpha}) > g(\bar{\alpha})$ or $h(\bar{\alpha}) > 0$.
3. Also note that $h(\bar{\alpha} = 1+e^{-1}) = 0.0975$.

By 1 - 3, we get that in the interval $1+e^{-1} \le \bar{\alpha} \le 2$, the point $(\bar{\alpha}, h(\bar{\alpha})) = (2,0)$ is an absolute minimum point. In other words, in the interval $1+e^{-1} < \bar{\alpha} < 2$, $h(\bar{\alpha}) > 0$ or $f(\bar{\alpha}) > g(\bar{\alpha})$. Q.E.D

Appendix B: The optimal logit CSF under discrimination

The designer controls the parameters δ and α. For $0 < \alpha \le 2$,[15] which ensures the existence of an interior equilibrium, see Konrad (2009), and for $\delta > 0$, the expected

[15] Notice that the proof in our case where $0 < \alpha \le 2$ is not a straightforward extension of the proof of Proposition 1 in Epstein et al. (2011) where $0 < \alpha \le 1$. When $0 < \alpha \le 1$, (6B) and (7B) are automatically satisfied and, in turn, the second order conditions are satisfied and the utility of every

payoff of the contestants are:

$$E(u_1)=\frac{n_1 x_1^{\alpha}}{x_1^{\alpha}+(\delta x_2)^{\alpha}}-x_1 \text{ and } E(u_2)=\frac{n_2(\delta x_2)^{\alpha}}{x_1^{\alpha}+(\delta x_2)^{\alpha}}-x_2 \tag{1.B}$$

The first order conditions are:

$$\frac{\partial E(u_1)}{\partial x_1}=\frac{\alpha x_1^{\alpha-1} n_1(\delta x_2)^{\alpha}}{\left[x_1^{\alpha}+(\delta x_2)^{\alpha}\right]^2}-1=0 \text{ and } \frac{\partial E(u_2)}{\partial x_2}=\frac{\alpha \delta^{\alpha} x_2^{\alpha-1} n_2 x_1^{\alpha}}{\left[x_1^{\alpha}+(\delta x_2)^{\alpha}\right]^2}-1=0 \tag{2.B}$$

and, after rearranging, we get that in equilibrium:

$$x_1^{*}=\frac{\alpha n_1(\delta k)^{\alpha}}{\left(k^{\alpha}+\delta^{\alpha}\right)^2},\ x_2^{*}=\frac{\alpha n_2(\delta k)^{\alpha}}{\left(k^{\alpha}+\delta^{\alpha}\right)^2} \text{ and } x_1^{*}+x_2^{*}=\frac{\alpha(\delta k)^{\alpha}(n_1+n_2)}{\left(k^{\alpha}+\delta^{\alpha}\right)^2} \tag{3.B}$$

$$p_1=\frac{k^{\alpha}}{k^{\alpha}+\delta^{\alpha}} \text{ and } p_2=\frac{\delta^{\alpha}}{k^{\alpha}+\delta^{\alpha}} \tag{4.B}$$

$$E(u_1^{*})=\frac{n_1 k^{\alpha}\left[(1-\alpha)\delta^{\alpha}+k^{\alpha}\right]}{\left(k^{\alpha}+\delta^{\alpha}\right)^2} \text{ and } E(u_2^{*})=\frac{n_2 \delta^{\alpha}\left[(1-\alpha)k^{\alpha}+\delta^{\alpha}\right]}{\left(k^{\alpha}+\delta^{\alpha}\right)^2} \tag{5.B}$$

The second order equilibrium conditions (SOC) are:

$$\frac{\partial^2 E(u_1)}{\partial x_1^2}=\frac{\alpha n_1(\delta x_2)^{\alpha} x_1^{\alpha-2}\left\{(\alpha-1)(\delta x_2)^{\alpha}-(\alpha+1)x_1^{\alpha}\right\}}{\left[x_1^{\alpha}+(\delta x_2)^{\alpha}\right]^3}\leq 0$$

$$\frac{\partial^2 E(u_2)}{\partial x_2^2}=\frac{\alpha n_2 \delta^{\alpha} x_1^{\alpha} x_2^{\alpha-2}\left\{(\alpha-1)x_1^{\alpha}-(\alpha+1)(\delta x_2)^{\alpha}\right\}}{\left[x_1^{\alpha}+(\delta x_2)^{\alpha}\right]^3}\leq 0$$

In equilibrium we obtain that $\frac{x_1^{*}}{x_2^{*}}=k$, therefore the SOC can be written as:

$$(1-\alpha)\delta^{\alpha}+(\alpha+1)k^{\alpha}\geq 0 \text{ and } (1-\alpha)k^{\alpha}+(\alpha+1)\delta^{\alpha}\geq 0 \tag{6.B}$$

Also, in equilibrium, the expected contestants' payoffs must be non-negative, that is, $E(u_1^{*})\geq 0$ and $E(u_2^{*})\geq 0$, which requires (see (5.B)) that

$$(1-\alpha)\delta^{\alpha}+k^{\alpha}\geq 0 \text{ and } (1-\alpha)k^{\alpha}+\delta^{\alpha}\geq 0 \tag{7.B}$$

Note that the conditions in (7.B) ensure that the SOC in (6.B) are also satisfied.

Now, anticipating the equilibrium efforts of the contestants, the designer's objective is to maximize $x_1^{*}+x_2^{*}=\frac{\alpha(\delta k)^{\alpha}(n_1+n_2)}{\left(k^{\alpha}+\delta^{\alpha}\right)^2}$. For $0<\alpha\leq 2$, the designer's

---

contestant is not negative. However, when $1<\alpha\leq 2$, (6B) and (7B) are not automatically satisfied. Hence, in the current proof of Proposition 2, we have to take into account constraints that were not relevant when $0<\alpha\leq 1$, as pointed out in the sequel, see problem (8B).

problem is:

$$
\begin{aligned}
&MaxG_L = \frac{\alpha(\delta k)^{\alpha}(n_1+n_2)}{(k^{\alpha}+\delta^{\alpha})^2} \\
&s.t. \\
&(1-\alpha)\delta^{\alpha}+k^{\alpha}\geq 0 \\
&(1-\alpha)k^{\alpha}+\delta^{\alpha}\geq 0 \\
&\delta\geq 0
\end{aligned}
\tag{8.B}
$$

The first two constraints ensure that the SOC for the equilibrium strategies of the two contestants are satisfied and that their expected payoffs are non-negative. Let us temporarily ignore these conditions and solve the designer's unconstrained problem in two stages. In the first stage, α is considered as given and δ is computed from the first order condition for the (unconstrained) maximization of G_L:

$$
\frac{\partial G_L}{\partial \delta} = \frac{\alpha^2\delta^{\alpha-1}k^{\alpha}(n_1+n_2)(k^{\alpha}-\delta^{\alpha})}{(k^{\alpha}+\delta^{\alpha})^3} = 0 \tag{9.B}
$$

which yields

$$
\delta^*(\alpha) = k \tag{10.B}
$$

Substituting this value into $\frac{\partial^2 G_L}{\partial \delta^2}$, we get that the SOC of the designer's problem is satisfied. That is,

$$
\frac{\partial^2 G_L}{\partial \delta^2} = -\frac{\alpha^3(n_1+n_2)}{32k^{3\alpha+2}} < 0
$$

Furthermore, the optimal bias in (10.B) also satisfies all the constraints of problem (8.B). $\delta^*(\alpha)=k$ is therefore the optimal bias for any α, $0<\alpha\leq 2$. Substituting $\delta^*(\alpha)=k$ in equations (3.B)-(5.B), we get that $x_1^*=0.25\alpha n_1$, $x_2^*=0.25\alpha n_2$, $p_1=p_2=0.5$, $E(u_1^*)=0.25n_1(2-\alpha)$, $E(u_2^*)=0.25n_2(2-\alpha)$ and $G_L=0.25\alpha(n_1+n_2)$. This implies that $\alpha=2$ [16] is optimal for the designer yielding the maximal certain aggregate efforts of $G_L=0.5(n_1+n_2)$.

Q.E.D

[16] Notice that Nti (2004) has established that for the existence of a unique pure-strategy equilibrium when $k>1$ it must be the case that $\alpha<2$. In our case, we allow discrimination between the contestants. Therefore we obtain that even though $k>1$, in a unique pure-strategy equilibrium $\alpha=2$.